A Direct Multisearch Filter Method for Biobjective Optimization

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Outline

1. Introduction
2. Direct Multisearch Filter (DMS-Filter)
3. Convergence Results
4. Computational Results
5. Conclusions and Future Work

Outline

Multiobjective Optimization

$$\min_{x \in \Omega \subseteq \mathbb{R}^n} \quad F(x) = \left(f_1(x), f_2(x), \ldots, f_m(x)\right)^\top$$

$$f_j : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, \ j = 1, 2, \ldots, m \geq 2$$

- $\Omega = X \cap \{x \in \mathbb{R}^n \mid C(x) \leq 0\}$ where X is a full dimensional polyhedron and $C : \mathbb{R}^n \to (\mathbb{R} \cup \{+\infty\})^p$
- objectives often conflicting
- expensive function evaluation
- impossible to use or approximate derivatives

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Motivation

- DMS → A. L. Custódio, J. F. A. Madeira, A. I. F. Vaz, and L. N. Vicente. *Direct multisearch for multiobjective optimization*, SIAM J. Optim. (2011), 21, 1109-1140
  - DMS_{dense} → Asymptotically dense in the unit sphere
  - $\mathrm{DMS}_{\oplus}$ → Coordinate directions

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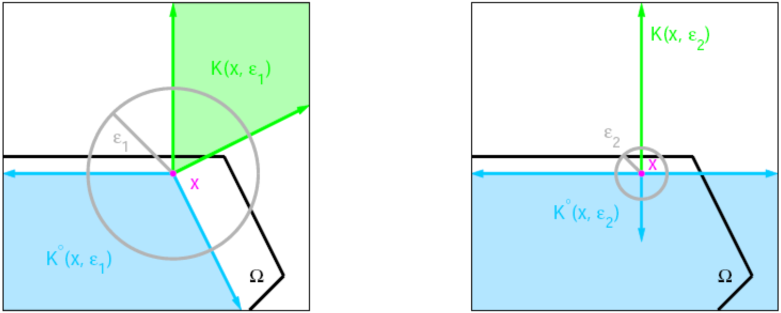
- Set of poll directions conforms to the geometry of nearby constraints
- Approach of Abramson, Brezhneva, Dennis, and Pingel [2008] for single objective optimization



(in Kolda, Lewis, and Torczon [2003])

Metrics for Performance Profiles (Dolan and Moré [2002])

- Purity

$$\frac{|F_{p,s} \cap F_p|}{|F_{p,s}|}$$

- Spreads Γ and Δ

$$\Gamma_{p,s} = \max_{j \in \{1,\ldots,m\}} \left(\max_{i \in \{0,\ldots,N\}} \{d_i\} \right)$$

$$\Delta = \max_{j \in \{1,\ldots,m\}} \left(\frac{d_0 + d_N + \sum_{i=1}^{N-1} |d_i - \bar{d}|}{d_0 + d_N + (N-1)\bar{d}} \right)$$

- Hypervolume

$$HI_{p,s} = Vol\{b \in \mathbb{R}^m \mid b \leq U_p \wedge \exists a \in F_{p,s} : a \leq b\}$$

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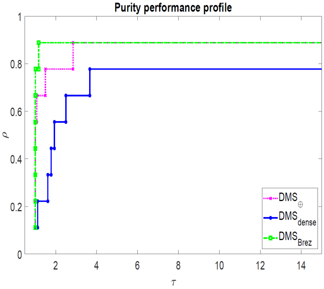
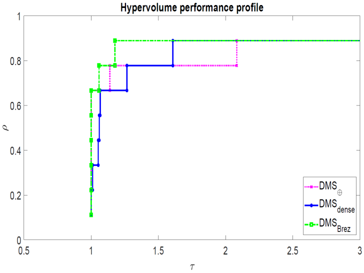

Purity performance profile
ρ
τ
DMS⊕
DMS dense
DMS Brez


Hypervolume performance profile
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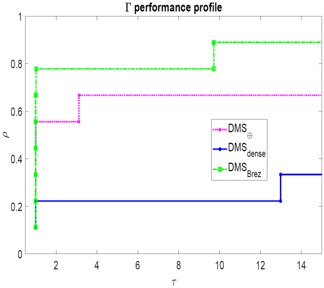
Γ performance profile
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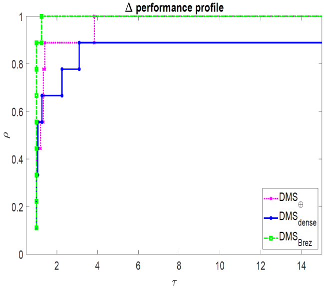
Δ performance profile
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- DFMO → G. Liuzzi, S. Lucidi, and F. Rinaldi. *A derivative-free approach to constrained multiobjective nonsmooth optimization*. SIAM J. Optim. (2016), 26, 2744-2774

- 93 biobjective problems with nonlinear constraints and bounds
  - number of variables between 3 and 30
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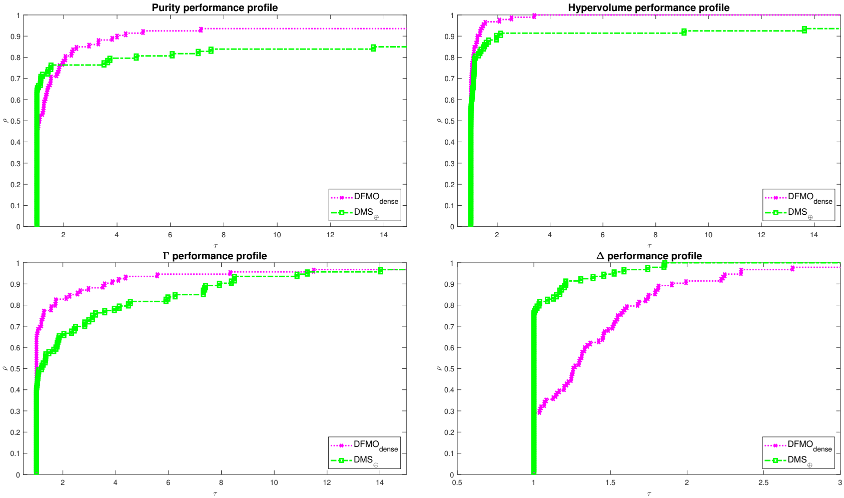
DMS - Nonlinear + Bound Constraints



DFMO

DMS

New Problem

- Extreme Barrier Function:

$$F_X(x) = \begin{cases} F(x), \text{ if } x \in X \\ (+\infty, +\infty, \dots, +\infty)^\top, \text{ otherwise} \end{cases}$$

- Constraint Violation function:

$$h(x) = \|C(x)_+\|_2^2 = \sum_{i=1}^{p} \max\{0, c_i(x)\}^2$$

$$\min_{x \in X} \left(f_1(x), f_2(x), \dots, f_m(x), h(x)\right)^\top$$

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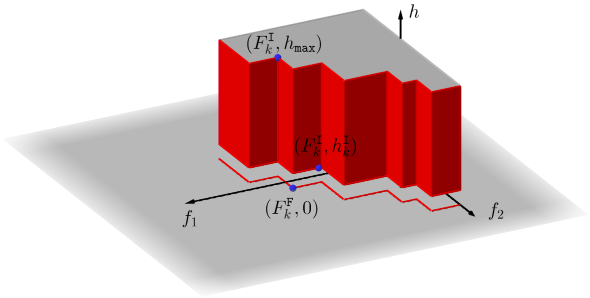
Filter Approach

The filter $\mathcal{F}$ is a set of nondominated points



A point x' is said to be filtered by a filter $\mathcal{F}$ if any of the following properties hold:

- There exists a point $x \in \mathcal{F}$ such that $x' \geq x$
- $h(x') > h_{\max}$ for some positive finite upper bound $h_{\max}$

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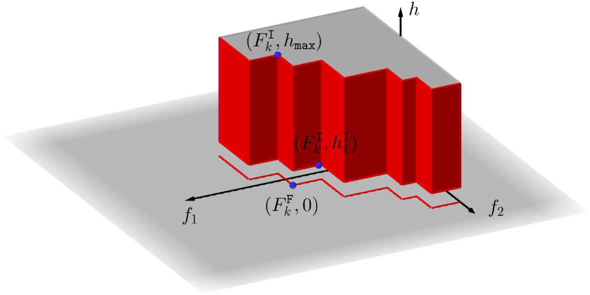

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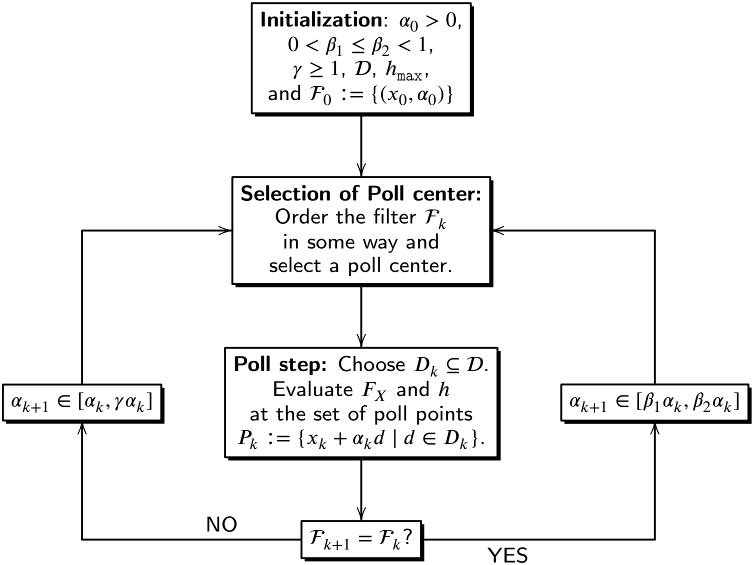
DMS-Filter - Algorithmic Structure



Solutions: $L := \{(x, \alpha) \in \mathcal{F} \mid (F_X(x), h(x)) = (F(x), 0)\}$.

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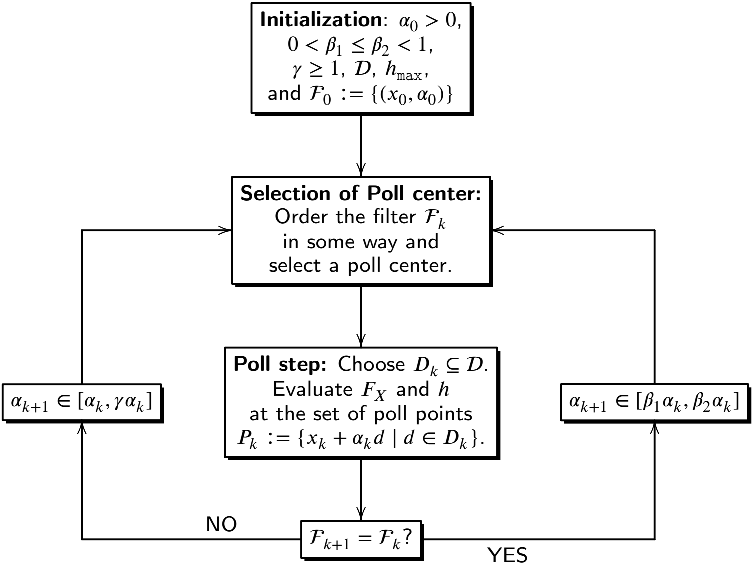

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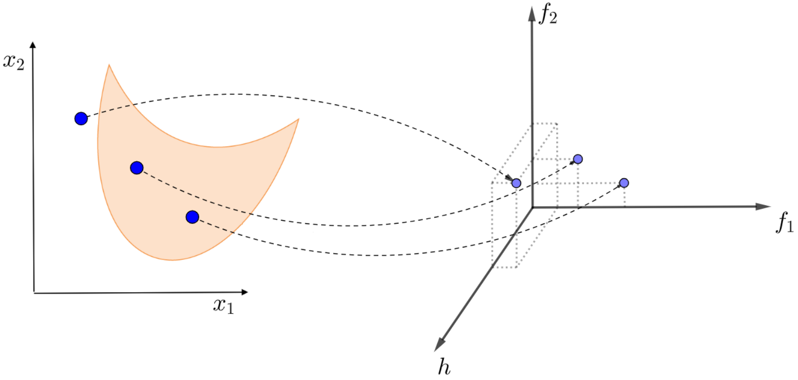
DMS-Filter – Poll Step



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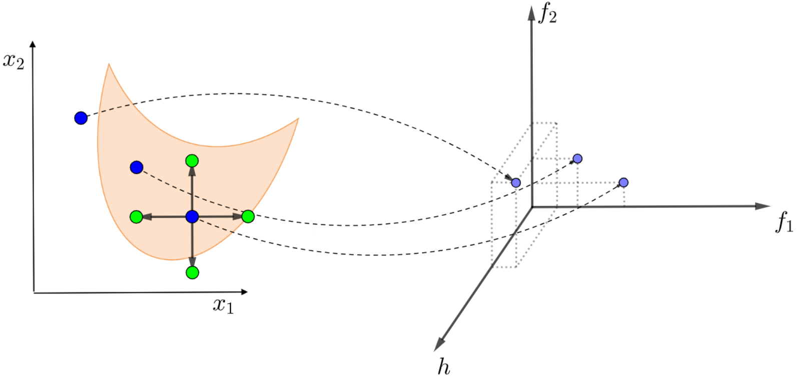

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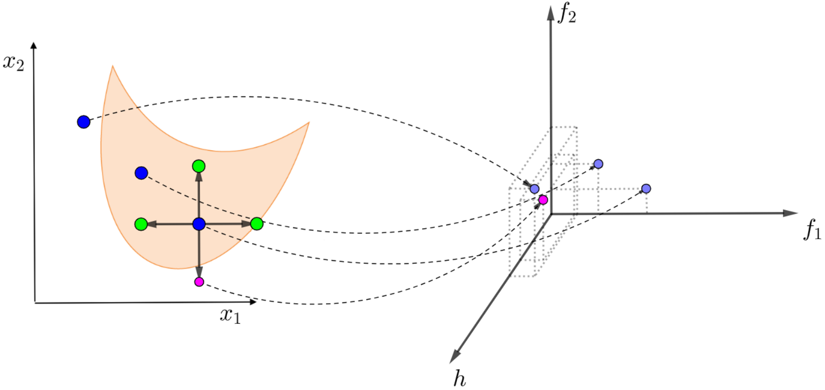

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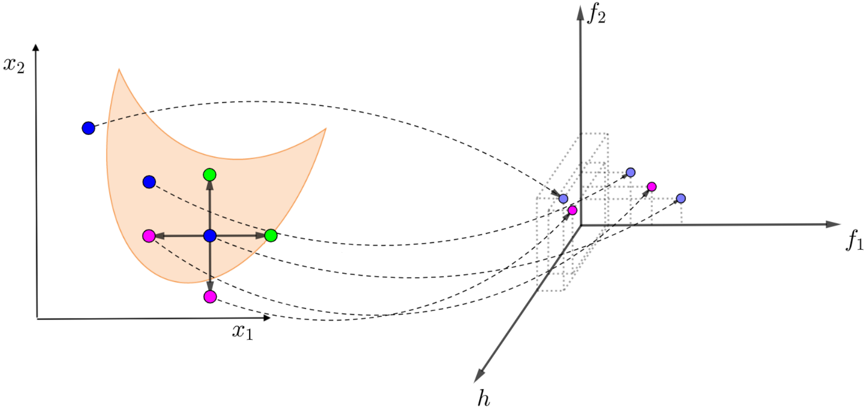

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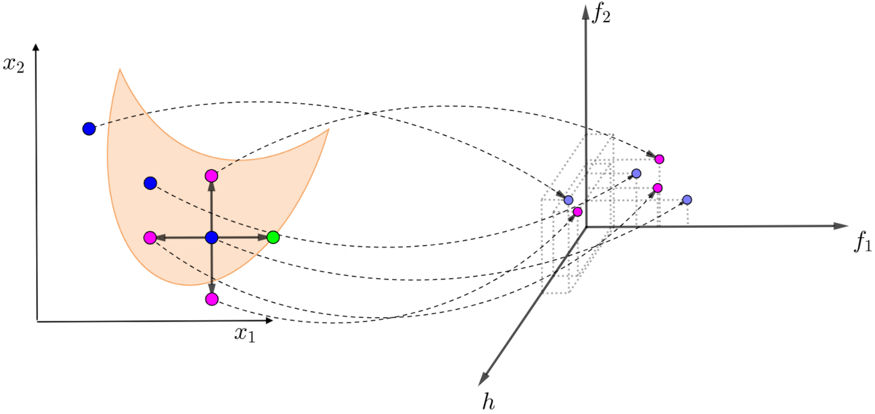

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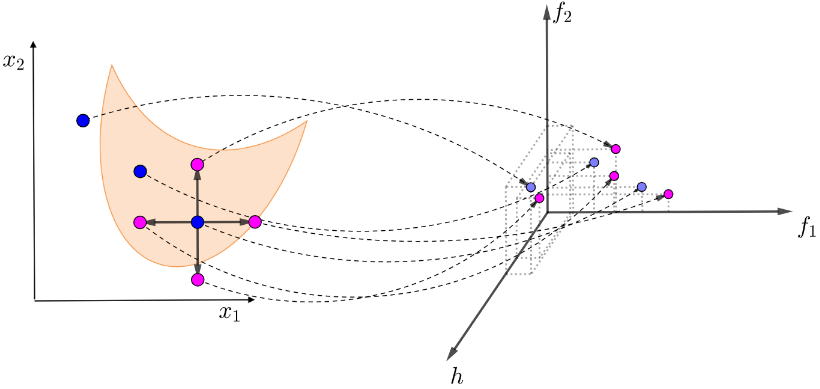

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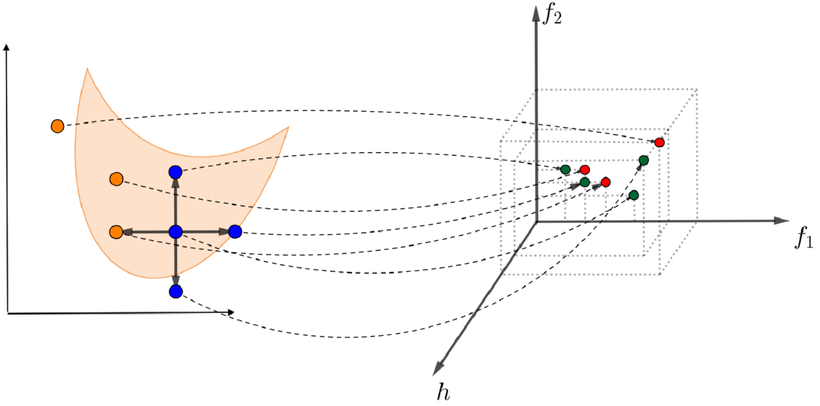

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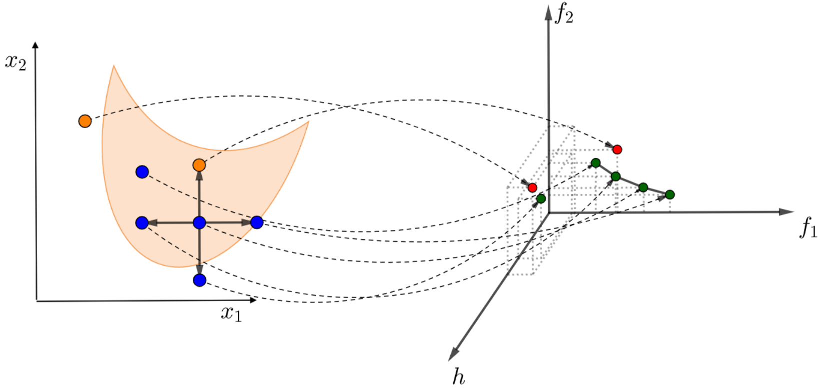
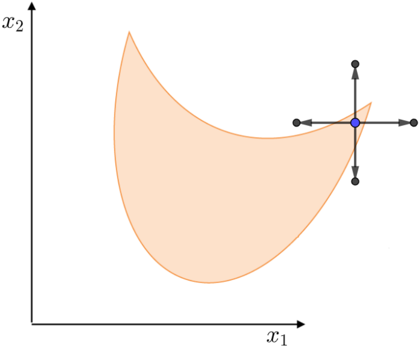
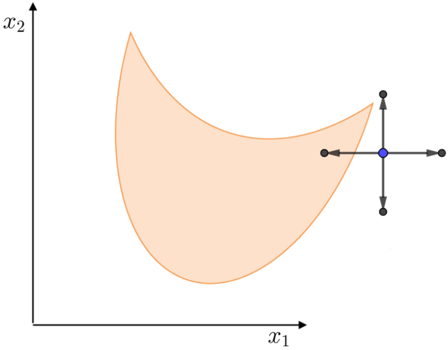


Poll Center Selection

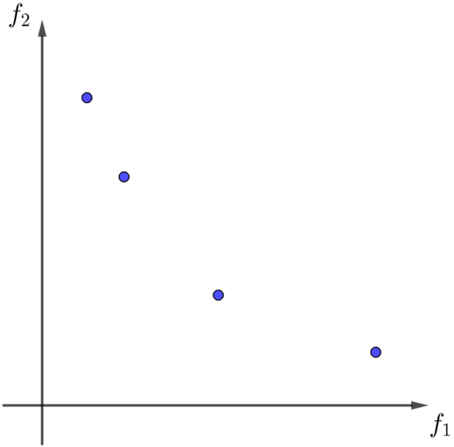
- **Feasible to Infeasible**



- **Infeasible to Feasible**



Feasible poll center - Most Isolated Point



- $\mathcal{F} \cap \Omega$

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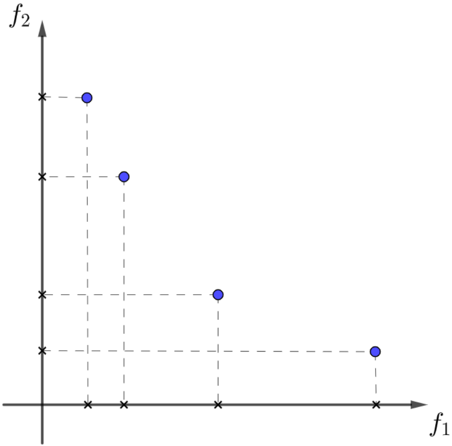

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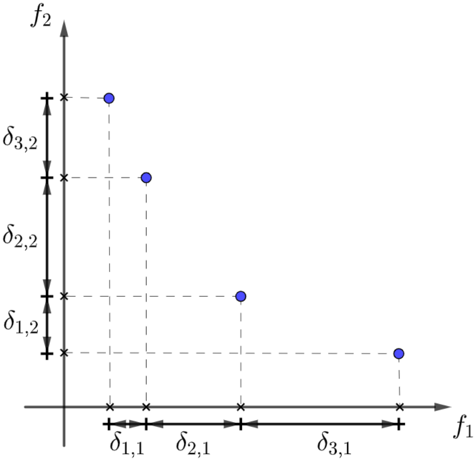
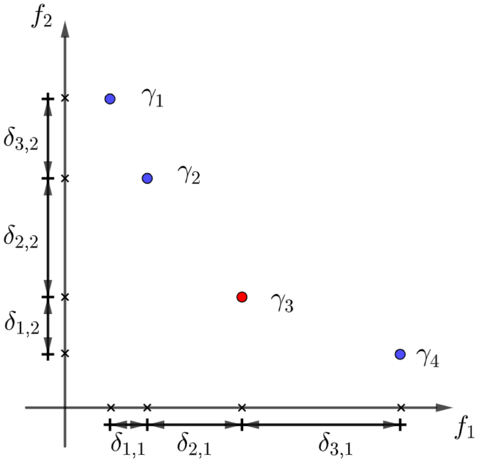


$\delta_{i,j} = f_{i+1,j} - f_{i,j}$
for $i = 1, 2, 3$ and $j = 1, 2$.

Feasible poll center - Most Isolated Point



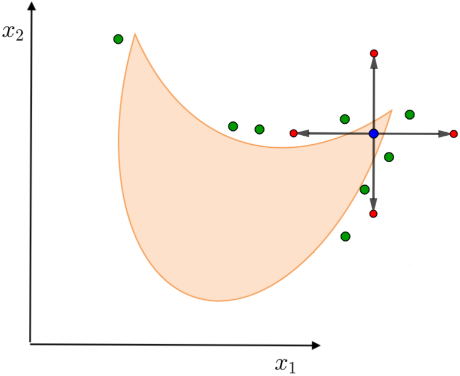
$$\gamma_1 = \frac{\delta_{3,2} + \delta_{1,1}}{2}$$

$$\gamma_2 = \frac{\frac{\delta_{3,2}+\delta_{2,2}}{2} + \frac{\delta_{1,1}+\delta_{2,1}}{2}}{2}$$

$$\gamma_3 = \frac{\frac{\delta_{2,2}+\delta_{1,2}}{2} + \frac{\delta_{2,1}+\delta_{3,1}}{2}}{2}$$

$$\gamma_4 = \frac{\delta_{1,2} + \delta_{3,1}}{2}$$

Infeasible poll center



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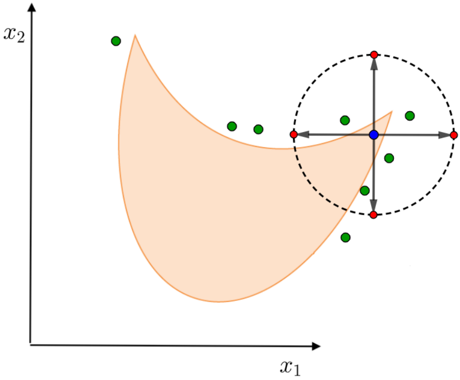

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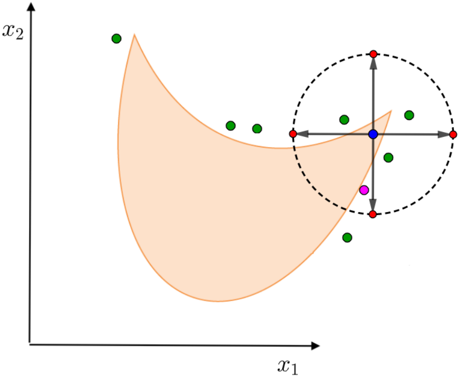

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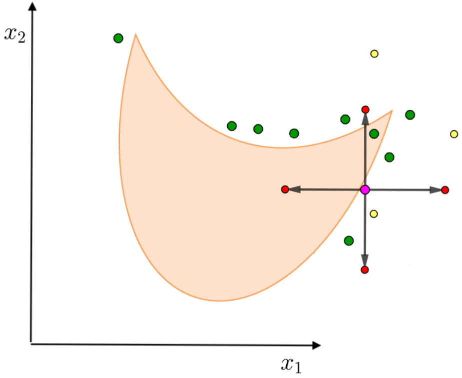


Outline

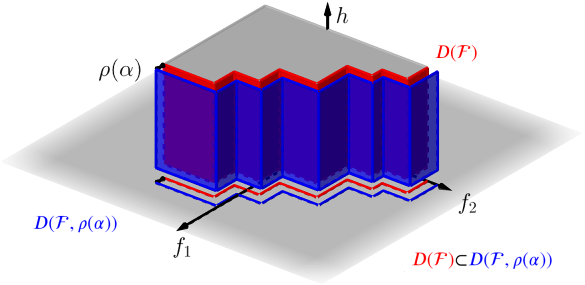
Globalization Strategies

Using Integer Lattices (Torczon [1997], Audet and Dennis [2002])

- requires only simple decrease
- poll directions and step size must satisfy integer/rational requirements

Imposing Sufficient Decrease (Kolda, Lewis, and Torczon [2003])

- use of a forcing function
  $\rho : (0, +\infty) \to (0, +\infty)$, continuous and nondecreasing, satisfying
  $\rho(t)/t \to 0$ when $t \downarrow 0$
- x is nondominated $\Leftrightarrow (F_X(x), h(x)) \notin D(\mathcal{F}, \rho(\alpha))$



Convergence Results

Theorem (Refining Subsequences)

There is at least a convergent refining subsequence of iterates $\{x_k\}_{k \in K}$, corresponding to unsuccessful poll steps, with $\lim_{k \in K} \alpha_k = 0$.

Let $\overline{x}$ be the limit point of a convergent refining subsequence $\{x_k\}_{k \in K}$.

Definition (Refining Directions)

Refining directions for $\overline{x}$ are limit points of $\{d_k / \|d_k\|\}_{k \in K}$, where $d_k \in D_k$ and $x_k + \alpha_k d_k \in S := \{x \in X \mid h(x) \leq h_{\max}\}$.

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Convergence Results

Assume that F and h are Lipschitz continuous near $\overline{x}$.

Theorem

- Let $\{x_k^{\mathrm{I}}\}_{k\in K}$ be an infeasible refining subsequence converging to $\overline{x} \in S$. If $d \in \mathrm{int}(T_S^{Cl}(\overline{x}))$ is a refining direction for $\overline{x}$ then:

$$h^{\circ}(\overline{x}; d) \geq 0$$

- Let $\{x_k^{\mathrm{F}}\}_{k\in K}$ be a feasible refining subsequence converging to $\overline{x} \in \Omega$. If $d \in \mathrm{int}(T_\Omega^{Cl}(\overline{x}))$ is a refining direction for $\overline{x}$ then:

$$\exists j = j(d) \in \{1, \dots, m\} \text{ such that } f_j^{\circ}(\overline{x}; d) \geq 0$$

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Numerical Settings

- Comparison among DFMO, DMS and DMS-Filter
- 93 biobjective problems with nonlinear constraints and bounds
  - number of variables between 3 and 30
  - number of constraints between 1 and 29
- Initialization with a feasible point
  - Feasible point provided by Karmitsa [2007]
- Initialization in line
  - n-points equally spaced in the line segment, joining the variable upper and lower bounds
- Stopping criterion
  - **DMS** and **DMS-Filter**:
    - $\alpha_k < 10^{-3}$ for all points in the filter
  - **DFMO**:
    - default values
  - maximum of 20000 function evaluations

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    - $\alpha_k < 10^{-3}$ for all points in the filter
  - **DFMO**:
    - default values
  - maximum of 20000 function evaluations

Numerical Settings

- Comparison among DFMO, DMS and DMS-Filter
- 93 biobjective problems with nonlinear constraints and bounds
  - number of variables between 3 and 30
  - number of constraints between 1 and 29
- Initialization with a feasible point
  - Feasible point provided by Karmitsa [2007]
- Initialization in line
  - n-points equally spaced in the line segment, joining the variable upper and lower bounds
- Stopping criterion
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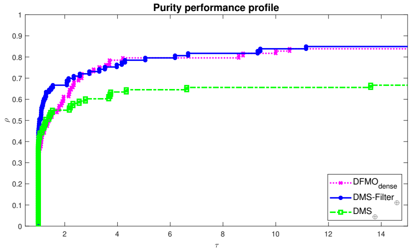
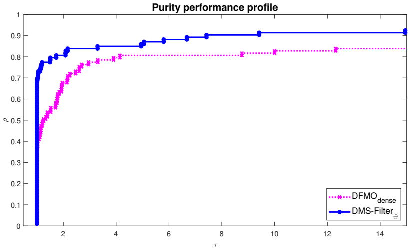
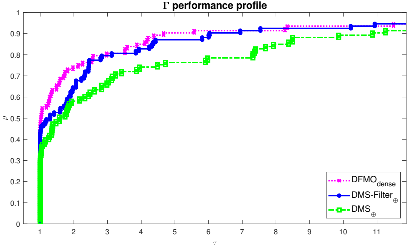
Results - Purity

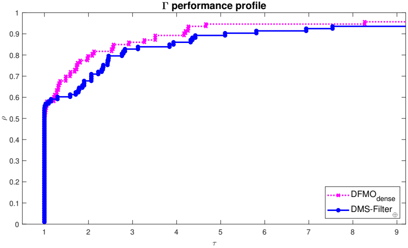




- DFMO
- DMS-Filter
- DMS

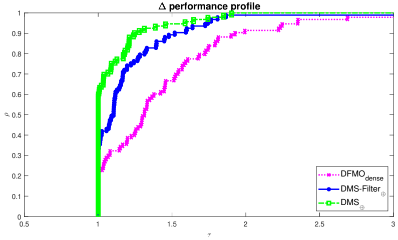
Results - Spread Gamma (Γ)

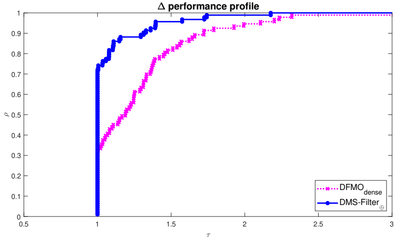




- DFMO
- DMS-Filter
- DMS

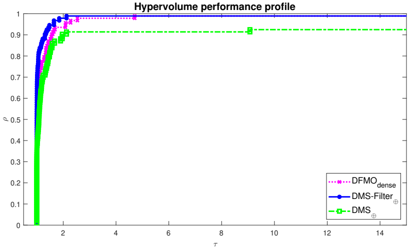
Results - Spread Delta (Δ)

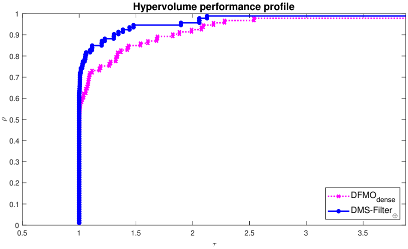




- DFMO
- DMS-Filter
- DMS

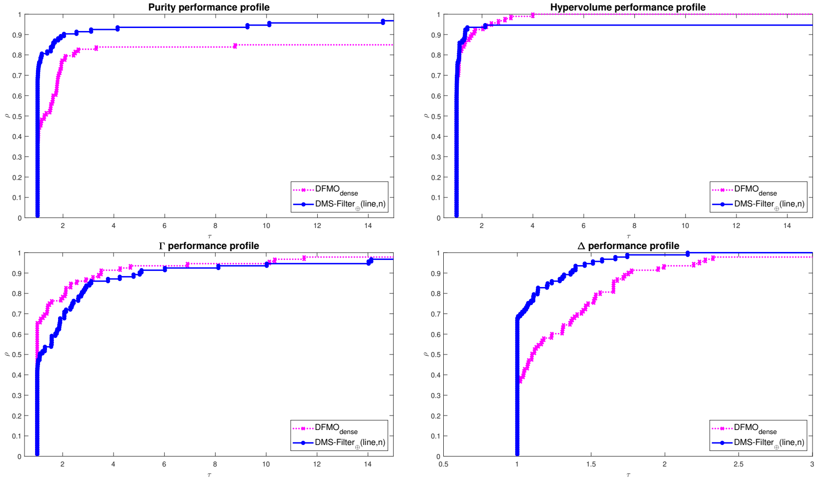
Results - Hypervolume





- DFMO
- DMS-Filter
- DMS

Results - DMS-Filter(line,n) VS DFMO



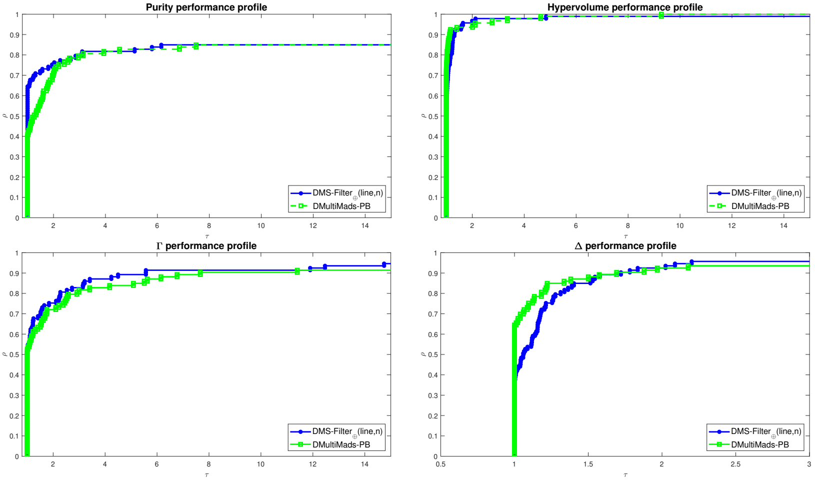
DFMO

DMS-Filter

Numerical Settings

- Comparison between DMS-Filter and DMultiMads-PB
  - **DMultiMads-PB** → Jean Bigeon, Sébastien Le Digabel, Ludovic Salomon. *Handling of constraints in multiobjective blackbox optimization*. ArXiv:2204.00904
- 93 biobjective problems with nonlinear constraints and bounds
  - number of variables between 3 and 30
  - number of constraints between 1 and 29
- Initialization in line
  - n-points equally spaced in the line segment, joining the variable upper and lower bounds
- Stopping criterion
  - **DMS-Filter** and **DMultiMads-PB**
    - $\alpha_k < 10^{-9}$ for one point
    - maximum of 20000 function evaluations

DMS-Filter VS DMultiMads-PB



DMS-Filter **DMultiMads-PB**

Outline

Conclusions and Future Work

- DMS-Filter extends filter methods to constrained Multiobjective Derivative-free Optimization
- DMS-Filter presents a well-supported convergence analysis for both globalization strategies
- DMS-Filter presents competitive numerical results for constrained Biobjective Derivative-free Optimization Problems

- Future work comprises extending the approach to problems with more than two objectives

THANKS FOR YOUR ATTENTION!

Any comments or questions?

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