A Direct Multisearch Filter Method for Biobjective Optimization

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Outline

1. Introduction
2. Direct Multisearch Filter (DMS-Filter)
3. Convergence Results
4. Computational Results
5. Conclusions and Future Work

Outline

Multiobjective Optimization

$$\min_{x \in \Omega \subseteq \mathbb{R}^n} \quad F(x) = \left(f_1(x), f_2(x), \dots, f_p(x)\right)^\top$$

$$f_j : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, \;\; j = 1, 2, \dots, p \geq 2$$

- $\Omega = X \cap \{x \in \mathbb{R}^n \mid C(x) \leq 0\}$ where X is a full dimensional polyhedron and $C : \mathbb{R}^n \to (\mathbb{R} \cup \{+\infty\})^m$
- objectives often conflicting
- expensive function evaluation
- impossible to use or approximate derivatives

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  - DMS_{dense} → Asymptotically dense in the unit sphere
  - $\mathrm{DMS}_{\oplus}$ → Coordinate directions

Motivation

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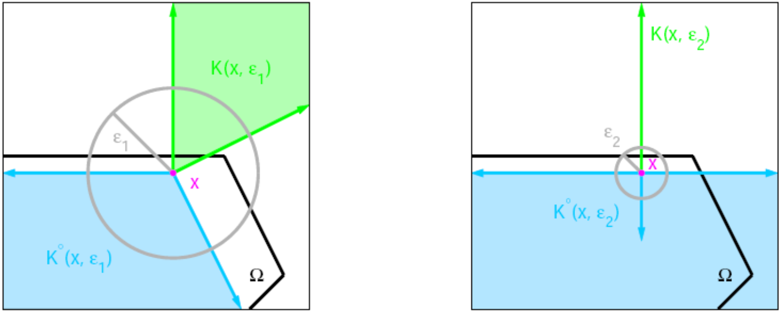
DMS - General Linear Constraints

- Set of poll directions conforms to the geometry of nearby constraints
- Proposed for single objective optimization in Abramson, Brezhneva, Dennis, and Pingel [2008].



(in Kolda, Lewis, and Torczon [2003])

Metrics for Performance Profiles (Dolan and Moré [2002])

- Purity

$$\frac{|F_{p,s} \cap F_p|}{|F_{p,s}|}$$

- Spreads Γ and Δ

$$\Gamma_{p,s} = \max_{j \in \{1,\ldots,m\}} \left(\max_{i \in \{0,\ldots,N\}} \{d_i\} \right)$$

$$\Delta = \max_{j \in \{1,\ldots,m\}} \left(\frac{d_0 + d_N + \sum_{i=1}^{N-1} |d_i - \bar{d}|}{d_0 + d_N + (N-1)\bar{d}} \right)$$

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$$HI_{p,s} = Vol\{b \in \mathbb{R}^m \mid b \le U_p \wedge \exists a \in F_{p,s} : a \le b\}$$

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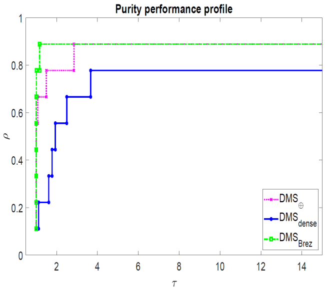
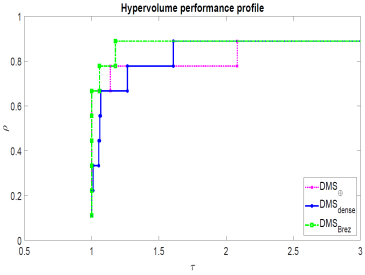

Purity performance profile
ρ
τ
DMS⊕
DMS dense
DMS Brez


Hypervolume performance profile
ρ
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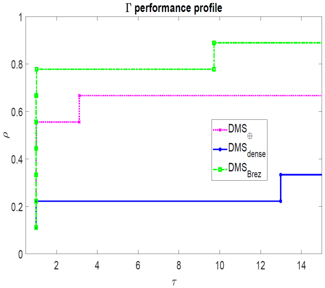
Γ performance profile
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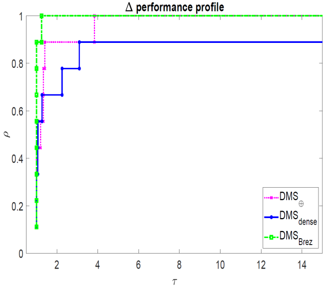
Δ performance profile
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- 93 biobjective problems with nonlinear constraints and bounds
  - number of variables between 3 and 30
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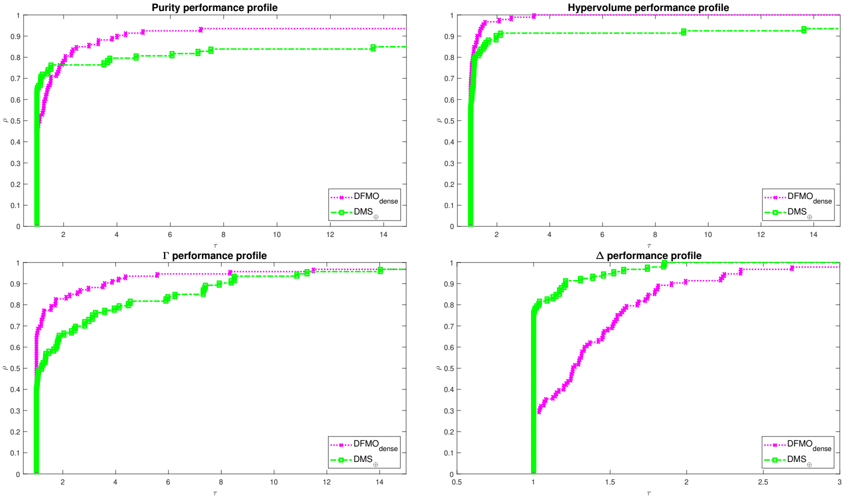
DMS - Nonlinear + Bound Constraints



-DFMO -DMS

New Problem

- Extreme Barrier Function:

$$F_X(x) = \begin{cases} F(x), \text{ if } x \in X \\ (+\infty, +\infty, \ldots, +\infty)^\top, \text{ otherwise} \end{cases}$$

- Constraint Violation function:

$$h(x) = \|C(x)_+\|_2^2 = \sum_{i=1}^{m} \max\{0, c_i(x)\}^2$$

$$\min_{x \in X} \left(f_1(x), f_2(x), \ldots, f_p(x), h(x)\right)^\top$$

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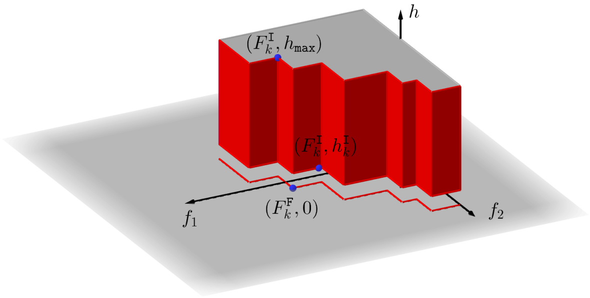
Filter Approach

The filter $\mathcal{F}$ is a set of nondominated points



A point x' is said to be filtered by a filter $\mathcal{F}$ if any of the following properties hold:

- There exists a point $x \in \mathcal{F}$ such that $x' \geq x$;
- $h(x') > h_{\max}$ for some positive finite upper bound $h_{\max}$

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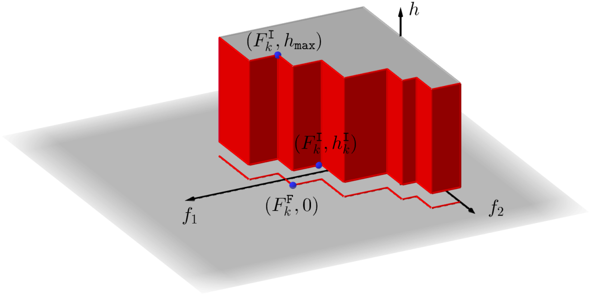

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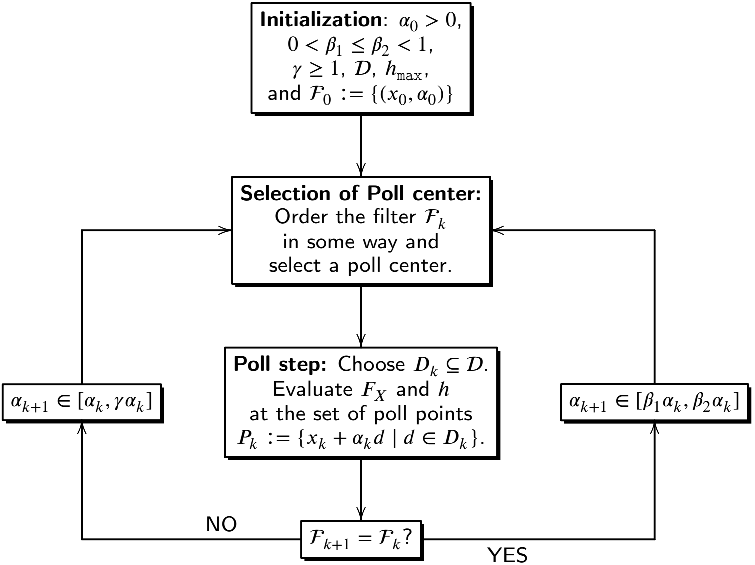
Outline

DMS-Filter - Algorithmic Structure



Solutions: $L := \{(x, \alpha) \in \mathcal{F} \mid (F_X(x), h(x)) = (F(x), 0)\}$.

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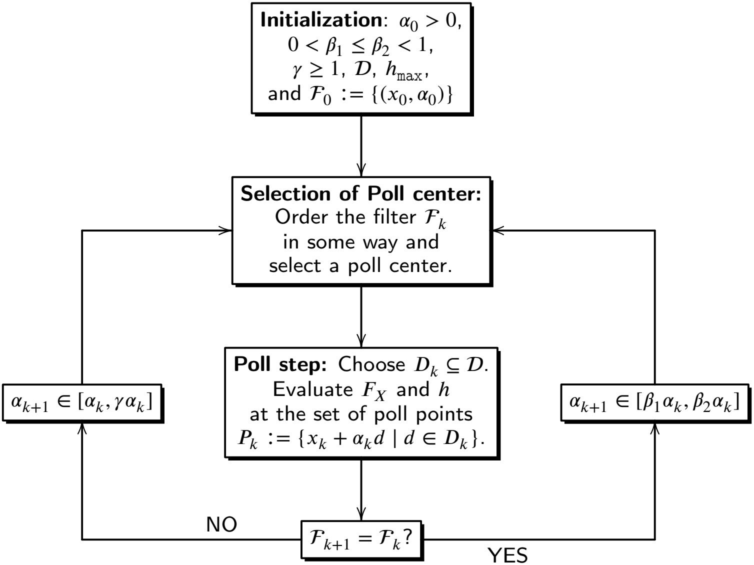

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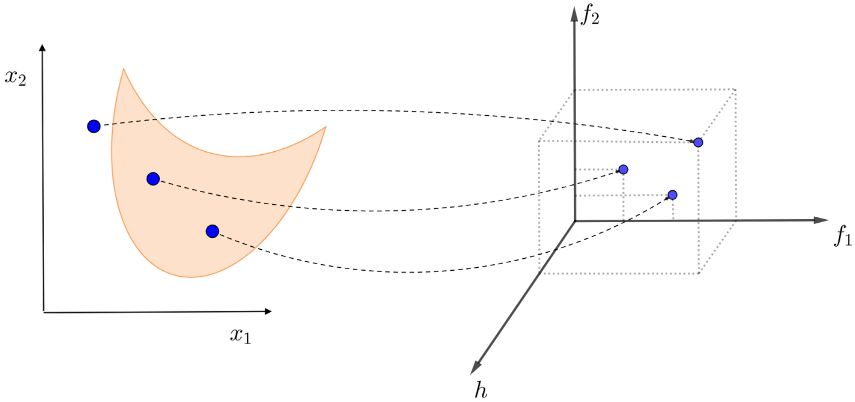
DMS-Filter – Poll Step



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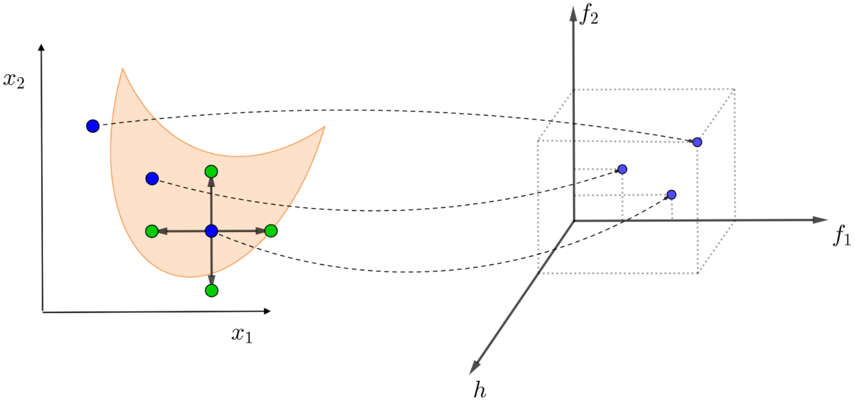

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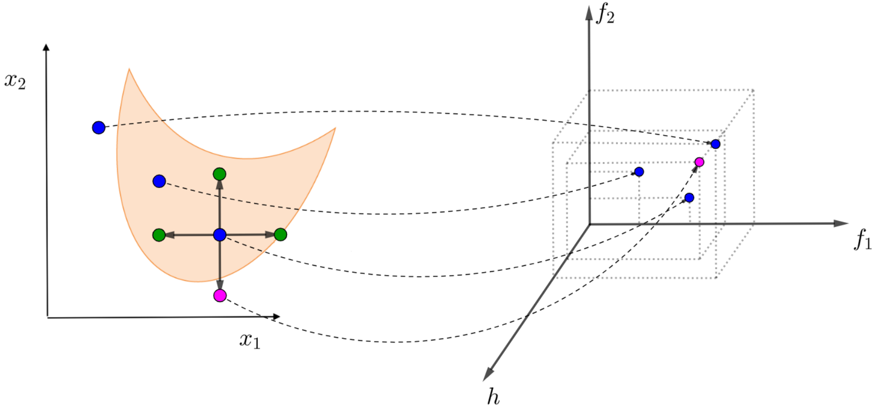

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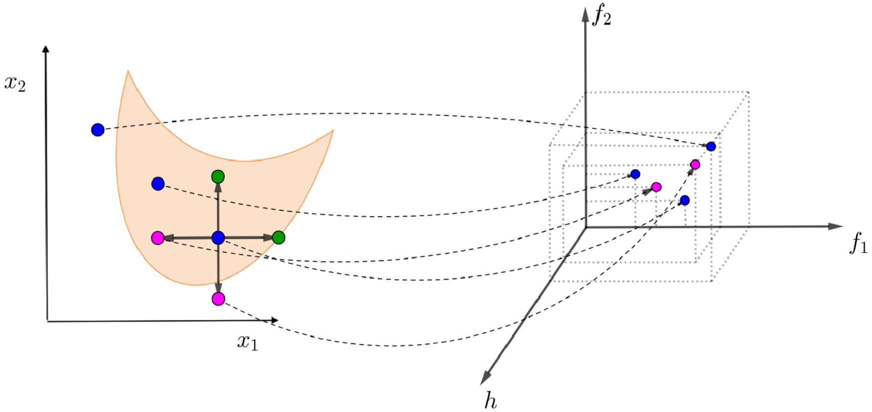

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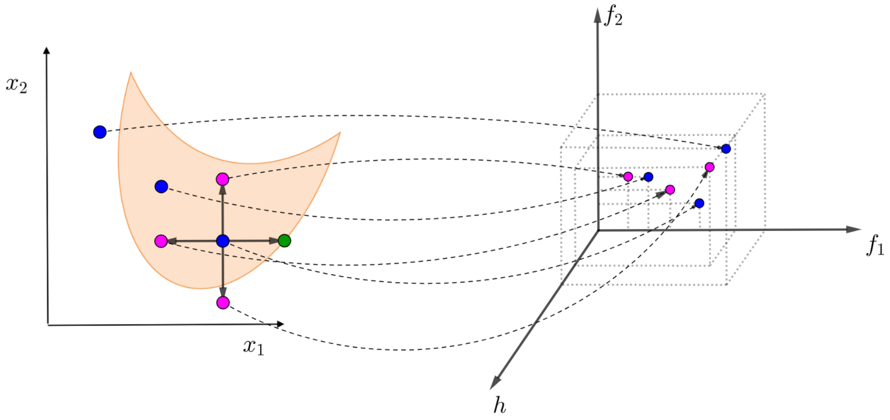

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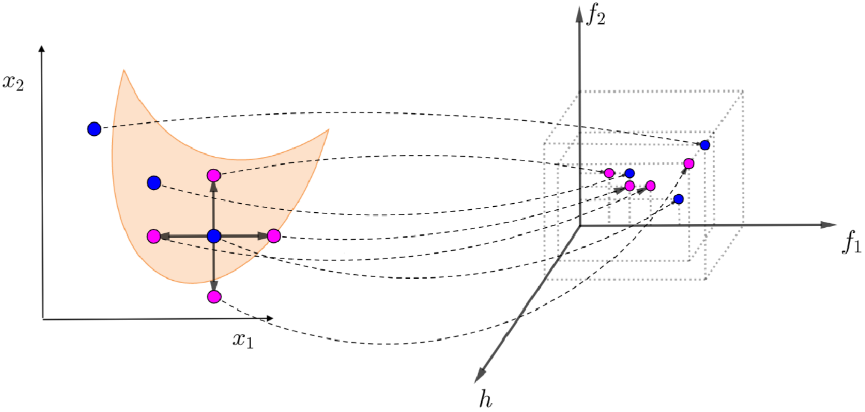

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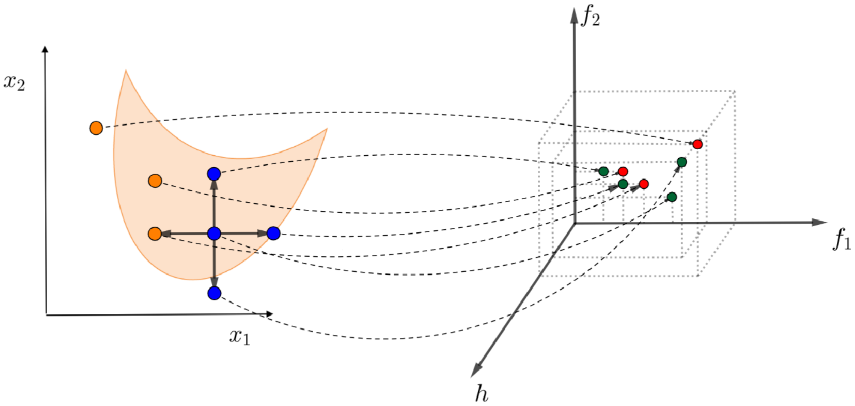

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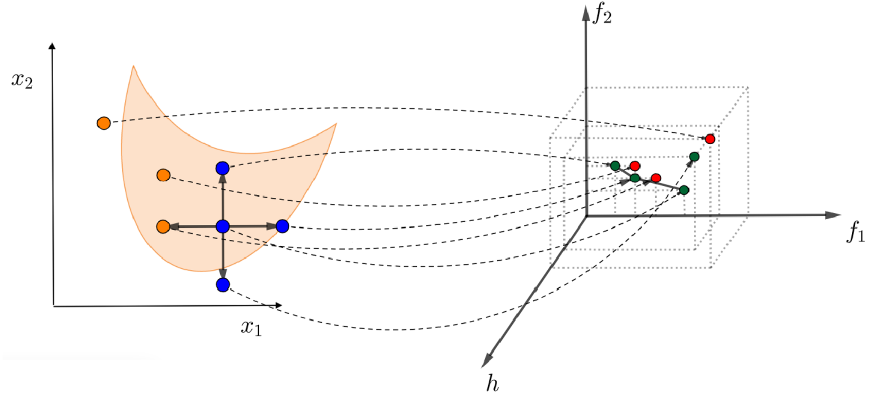
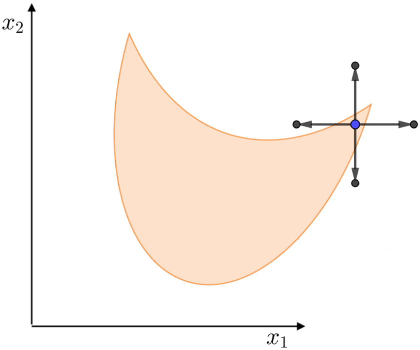
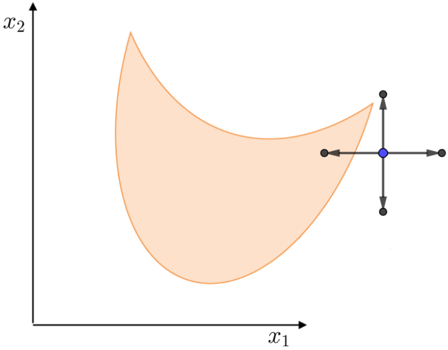


Poll Center Selection

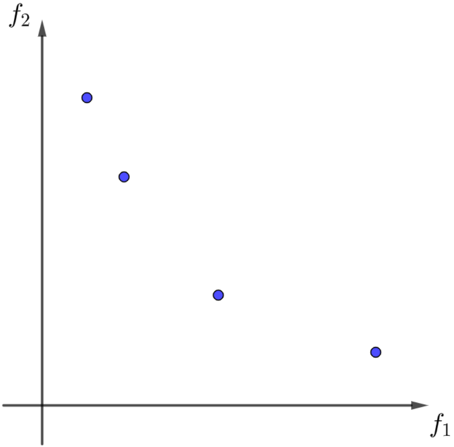
- **Feasible to Infeasible**



- **Infeasible to Feasible**

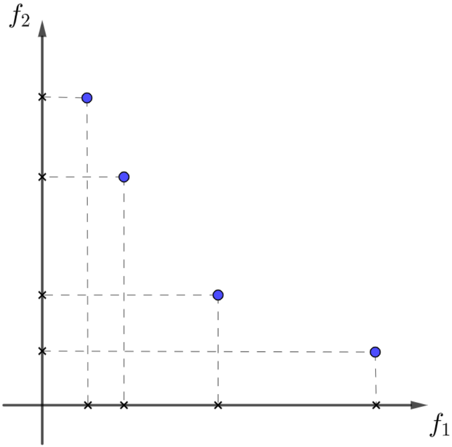


Feasible poll center - Most Isolated Point



- $\mathcal{F} \cap \Omega$

Feasible poll center - Most Isolated Point



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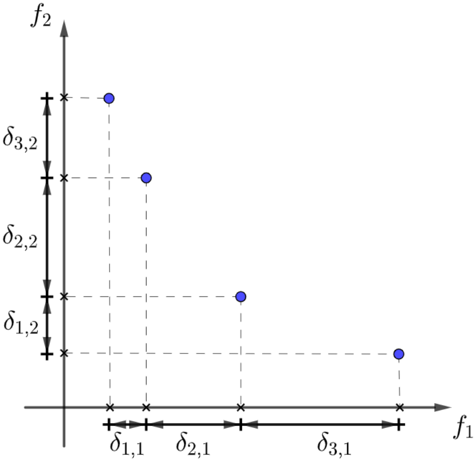
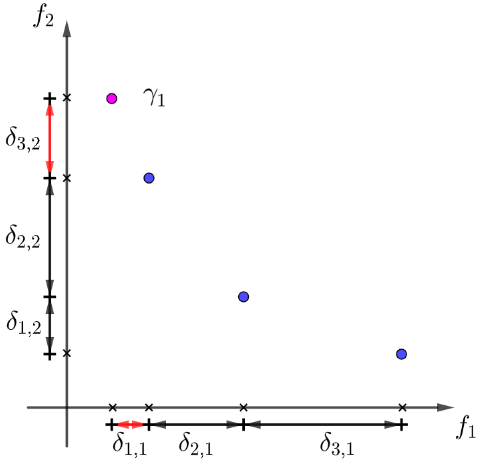


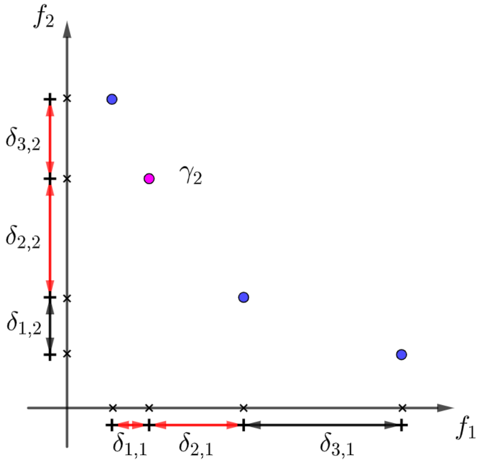
$\delta_{i,j} = f_{i+1,j} - f_{i,j}$
for $i = 1, 2, 3$ and $j = 1, 2$.

Feasible poll center - Most Isolated Point



$$\gamma_1 = \frac{\delta_{3,2} + \delta_{1,1}}{2}$$

Feasible poll center - Most Isolated Point



$$\gamma_2 = \frac{\frac{\delta_{3,2}+\delta_{2,2}}{2} + \frac{\delta_{1,1}+\delta_{2,1}}{2}}{2}$$

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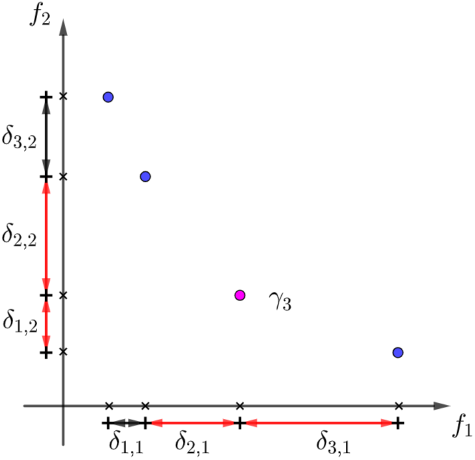
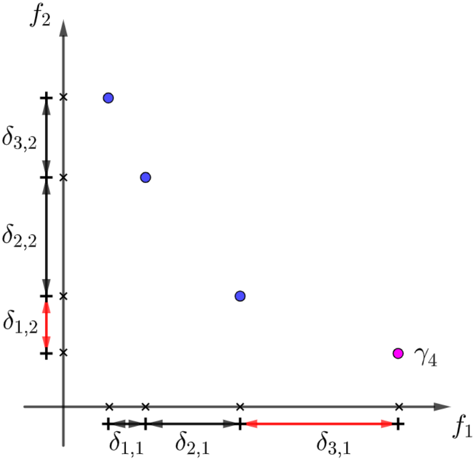


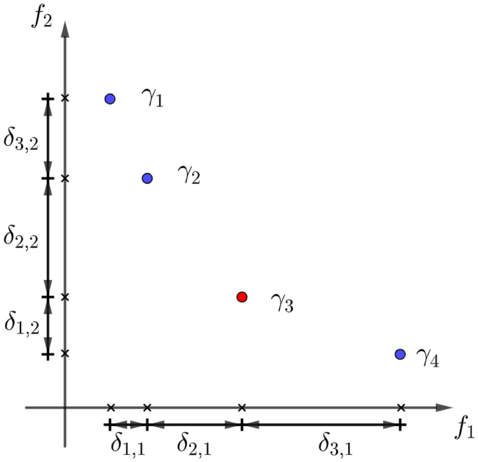
$$\gamma_3 = \frac{\frac{\delta_{2,2}+\delta_{1,2}}{2} + \frac{\delta_{2,1}+\delta_{3,1}}{2}}{2}$$

Feasible poll center - Most Isolated Point



$$\gamma_4 = \frac{\delta_{1,2} + \delta_{3,1}}{2}$$

Feasible poll center - Most Isolated Point



$$\gamma_1 = \frac{\delta_{3,2} + \delta_{1,1}}{2}$$

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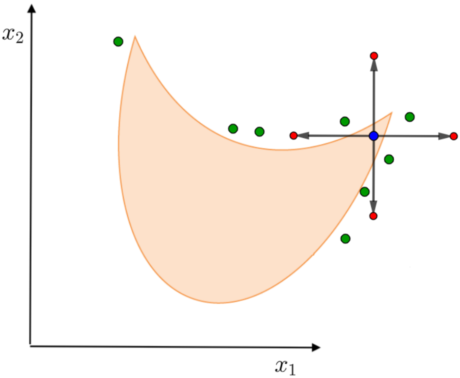
Infeasible poll center



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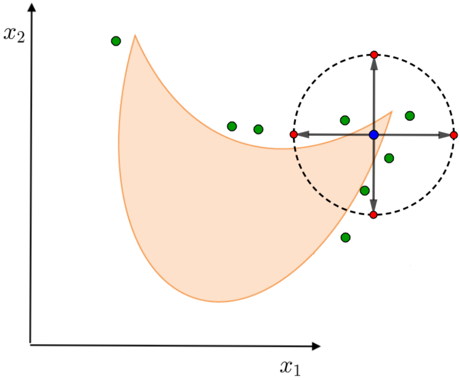

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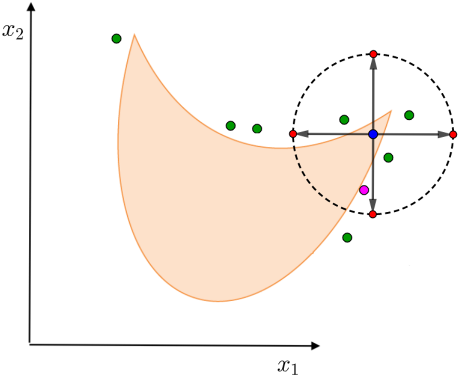

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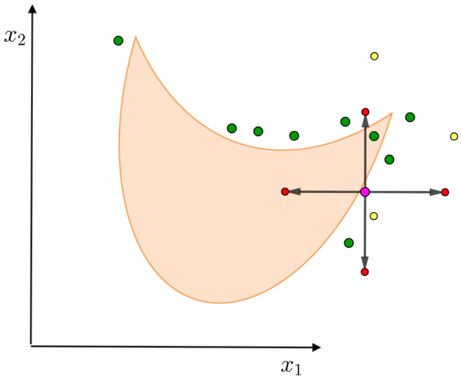


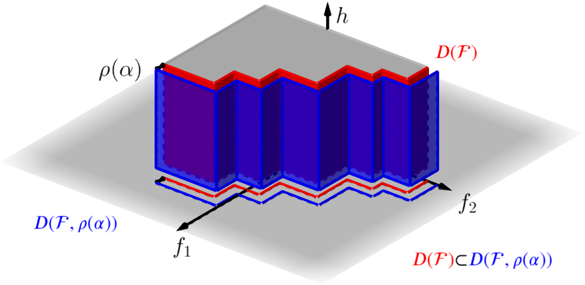
Outline

Using Integer Lattices (Torczon [1997], Audet and Dennis [2002])

- requires only simple decrease
- poll directions and step size must satisfy integer/rational requirements

Imposing Sufficient Decrease (Kolda, Lewis, and Torczon [2003])

- use of a forcing function
  $\rho : (0, +\infty) \rightarrow (0, +\infty)$, continuous and nondecreasing, satisfying
  $\rho(t)/t \rightarrow 0$ when $t \downarrow 0$
- x is nondominated $\Leftrightarrow (F_X(x), h(x)) \notin D(\mathcal{F}, \rho(\alpha))$



Convergence Results

Theorem (Refining Subsequences)

There is at least a convergent refining subsequence of iterates $\{x_k\}_{k\in K}$, corresponding to unsuccessful poll steps, with $\lim_{k\in K} \alpha_k = 0$.

Let $\overline{x}$ be the limit point of a convergent refining subsequence $\{x_k\}_{k\in K}$.

Definition (Refining Directions)

Refining directions for $\overline{x}$ are limit points of $\{d_k/\|d_k\|\}_{k\in K}$, where $d_k \in D_k$ and $x_k + \alpha_k d_k \in S := \{x \in X \mid h(x) \leq h_{\max}\}$

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Convergence Results

Assume that F and h are Lipschitz continuous near $\overline{x}$

Theorem

- $\{x_k^{\mathrm{I}}\}_{k\in K}$ an infeasible refining subsequence converging to $\overline{x} \in S$. If $d \in \mathrm{int}(T_S^{Cl}(\overline{x}))$ is a refining direction for $\overline{x}$ then:

$$h^\circ(\overline{x}; d) \geq 0$$

- $\{x_k^{\mathrm{F}}\}_{k\in K}$ a feasible refining subsequence converging to $\overline{x} \in \Omega$. If $d \in \mathrm{int}(T_\Omega^{Cl}(\overline{x}))$ is a refining direction for $\overline{x}$ then:

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Theorem

- $\{x_k^{\mathrm{I}}\}_{k\in K}$ an infeasible refining subsequence converging to $\overline{x} \in \mathcal{S}$. If the set of refining directions for $\overline{x}$ is dense in $\mathrm{int}(T_S^{Cl}(\overline{x})) \neq \emptyset$ then $\overline{x}$ is a Clarke critical point:

$$\forall d \in T_S^{Cl}(\overline{x}), h^{\circ}(\overline{x}; d) \geq 0$$

- $\{x_k^{\mathrm{F}}\}_{k\in K}$ a feasible refining subsequence converging to $\overline{x} \in \Omega$. If the set of refining directions for $\overline{x}$ is dense in $\mathrm{int}(T_\Omega^{Cl}(\overline{x})) \neq \emptyset$ then $\overline{x}$ is a Pareto-Clarke critical point:

$$\forall d \in T_\Omega^{Cl}(\overline{x}), \exists j = j(d) \in \{1, \dots, p\} \;\text{ such that }\; f_j^{\circ}(\overline{x}; d) \geq 0$$

Convergence Results

Theorem

- $\{x_k^{\mathrm{I}}\}_{k\in K}$ an infeasible refining subsequence converging to $\overline{x} \in \mathcal{S}$. If the set of refining directions for $\overline{x}$ is dense in $\mathrm{int}(T_S^{Cl}(\overline{x})) \neq \emptyset$ then $\overline{x}$ is a Clarke critical point:

$$\forall d \in T_S^{Cl}(\overline{x}), h^{\circ}(\overline{x}; d) \geq 0$$

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$$\forall d \in T_\Omega^{Cl}(\overline{x}), \exists j = j(d) \in \{1, \ldots, p\} \;\text{ such that }\; f_j^{\circ}(\overline{x}; d) \geq 0$$

Outline

Numerical Settings

- Comparison among DFMO, DMS and DMS-Filter
- 93 biobjective problems with nonlinear constraints and bounds
  - number of variables between 3 and 30
  - number of constraints between 1 and 29
- Initialization with a feasible point
  - Feasible point provided by Karmitsa [2007]
- Initialization in line
  - n-points equally spaced in the line segment, joining the variable upper and lower bounds
- Stopping criterion
  - **DMS** and **DMS-Filter**:
    - $\alpha_k < 10^{-3}$ for all points in the filter
  - **DFMO**:
    - default values
  - maximum of 20000 function evaluations

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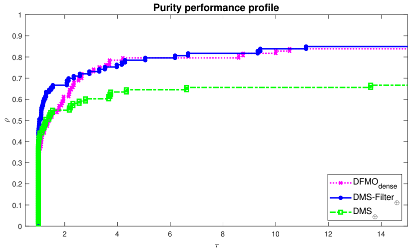
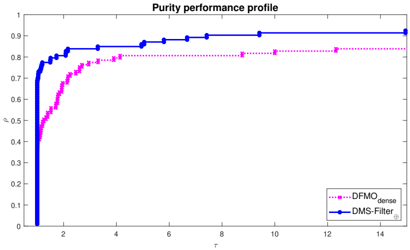
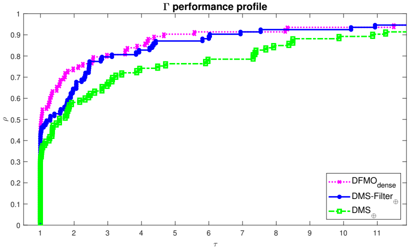
Results - Purity

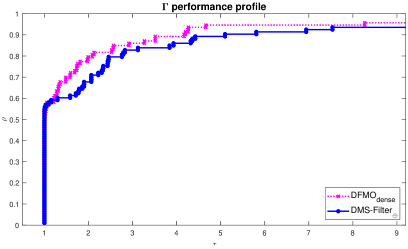




- DFMO
- DMS-Filter
- DMS

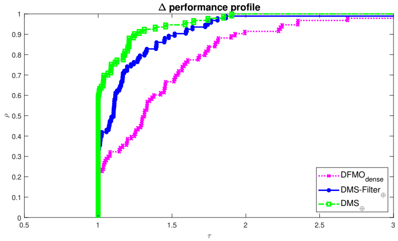
Results - Spread Gamma (Γ)

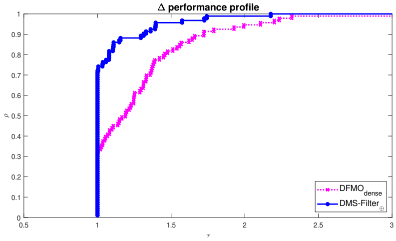




- DFMO
- DMS-Filter
- DMS

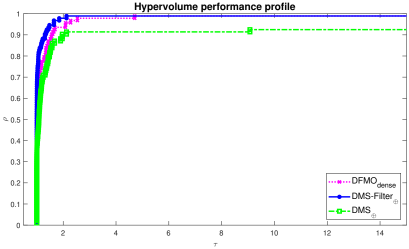
Results - Spread Delta (Δ)

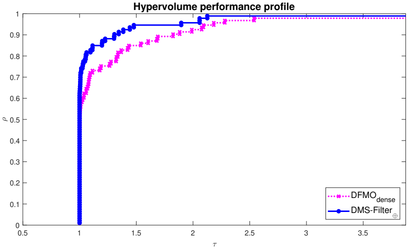




- DFMO
- DMS-Filter
- DMS

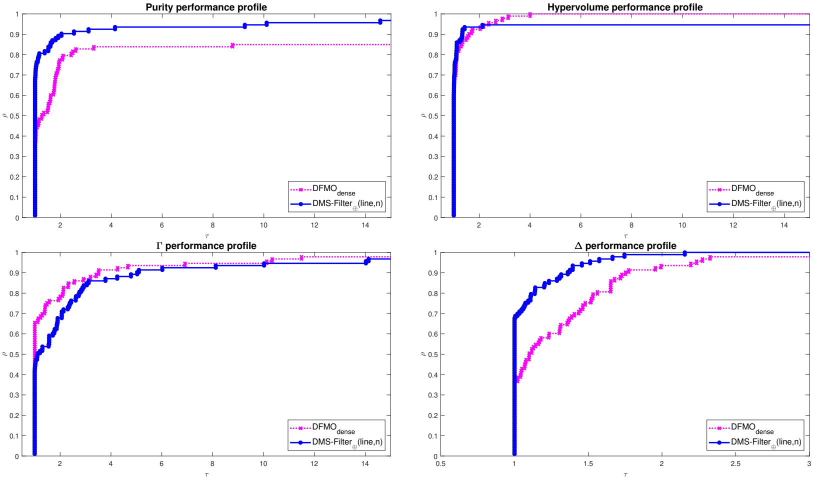
Results - Hypervolume





- DFMO
- DMS-Filter
- DMS

Results - DMS-Filter(line,n) VS DFMO



- DFMO
- DMS-Filter

Outline

Conclusions and Future Work

- DMS-Filter extends filter methods to constrained Multiobjective Derivative-free Optimization
- DMS-Filter presents a well-supported convergence analysis for both globalization strategies
- DMS-Filter presents competitive numerical results for constrained Biobjective Derivative-free Optimization Problems

- Future work comprises extending the approach to problems with more than two objectives

THANKS FOR YOUR ATTENTION!

Any comments or questions?

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