A Direct Multisearch Filter Method for Biobjective Optimization

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Optimization: Basic ideas

- x - Decision variable
- $f : \mathbb{R}^n \to \mathbb{R}$ - Objective Function
- S - Feasible set (requirements or constraints)

$$\begin{aligned} &\text{minimize } f(x) \\ &\text{subject to } \; x \in S \end{aligned} \qquad \text{(P)}$$

where $S := \{x \in X \subseteq \mathbb{R}^n \mid C(x) \leq 0\}$ with $C : \mathbb{R}^n \to (\mathbb{R} \cup \{+\infty\})^m$.

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Goal

Find $x^* \in S$ such that:

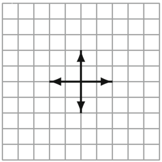
$$f(x^*) \leq f(x), \ \forall x \in S$$

Numerical Optimization

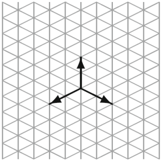
Iterative Methods

$$x_{k+1} = x_k + \alpha_k d_k$$

- Derivative-based methods: d_k is a descent direction such that $d_k^\top \nabla f(x_k) < 0$
- Derivative-free methods: when derivatives are not available and cannot be numerically approximated
  - Directional Direct Search - Sample function at positive spanning sets
    In $\mathbb{R}^2$:



$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$



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Problem Features

Multiobjective Optimization

$$\min_{x \in S \subseteq \mathbb{R}^n} \quad F(x) = \big(f_1(x), f_2(x), \dots, f_p(x)\big)^\top$$

$$f_j : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, \; j = 1, 2, \dots, p \geq 2$$

- objectives often conflicting
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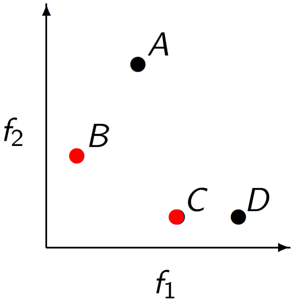
- objectives often conflicting
- expensive function evaluation
- impossible to use or approximate derivatives
- make use of Pareto Dominance

Pareto Dominance (x dominates y)

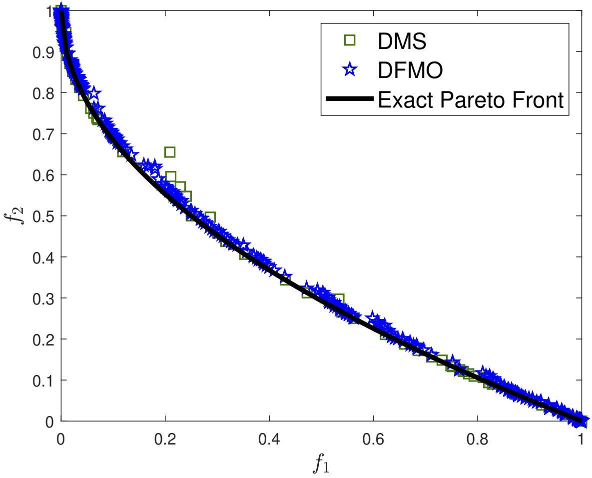
$F(x) \leq F(y)$, with $F(x) \neq F(y)$



Pareto Front: $\{B, C\}$

Motivation

- L1ZDT4 constrained Problem



□ DMS - Custódio, Madeira, Vaz and Vicente (2011)

☆ DFMO - Liuzzi, Lucidi, and Rinaldi (2016)

Defining a New Problem

Recall

$S = X \cap \{x \in \mathbb{R}^n \mid C(x) \leq 0\}$ where $C : \mathbb{R}^n \to (\mathbb{R} \cup \{+\infty\})^m$, and X a full dimensional polyhedron

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- Extreme Barrier Function:

$$F_X(x) = \begin{cases} F(x), & \text{if } x \in X \\ (+\infty, +\infty, \dots, +\infty)^\top, & \text{otherwise} \end{cases}$$

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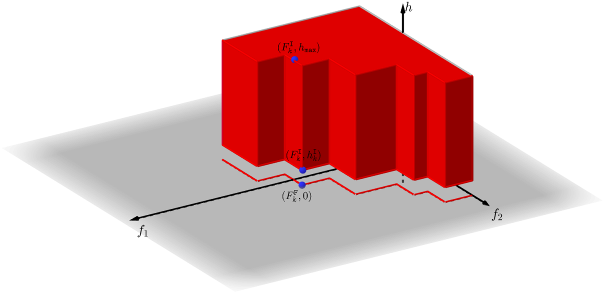
$$\min_{x \in X} \ \big(f_1(x), f_2(x), h(x)\big)^\top$$

Filter

The filter $\mathcal{F}$ is a set of nondominated points



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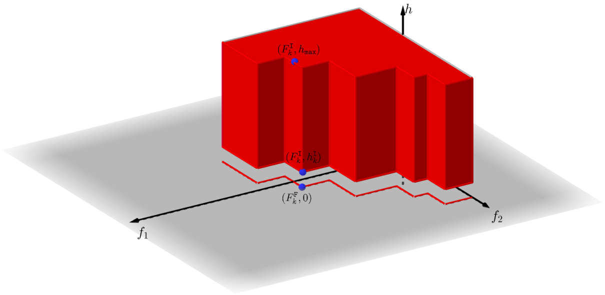
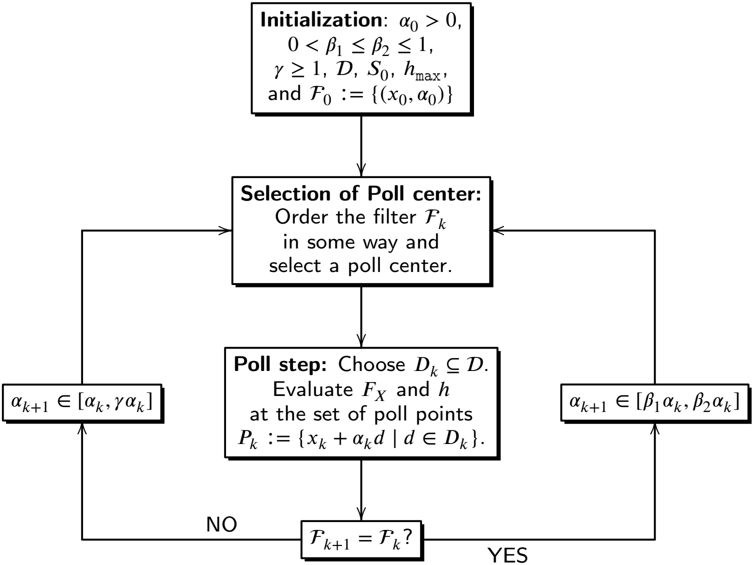


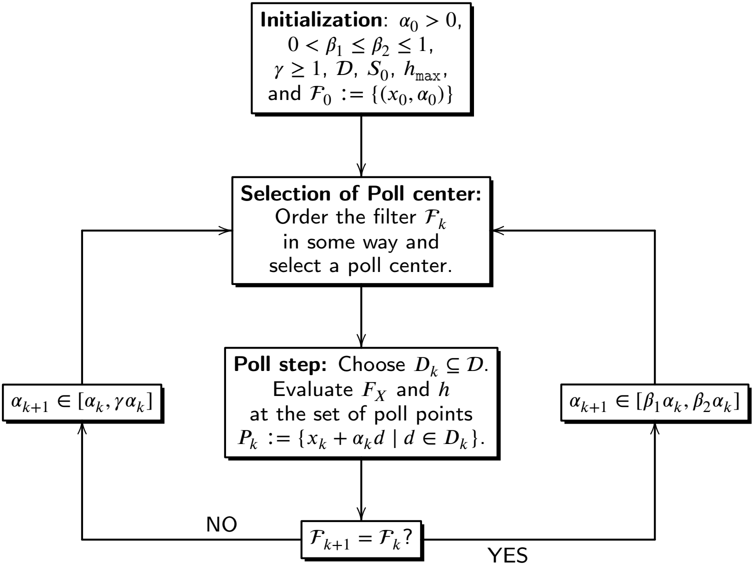
A point x' is said to be filtered by a filter $\mathcal{F}$ if any of the following properties hold:

- There exists a point $x \in \mathcal{F}$ such that $x' \geq x$;
- $h(x') > h_{\max}$ for some positive finite upper bound $h_{\max}$

DMS-Filter

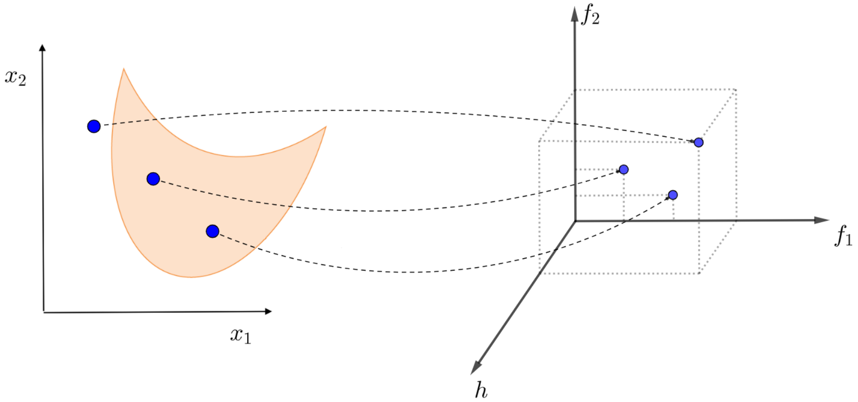

Initialization: $\alpha_0 > 0$, $0 < \beta_1 \leq \beta_2 \leq 1$, $\gamma \geq 1$, $\mathcal{D}$, S_0, $h_{\mathtt{max}}$, and $\mathcal{F}_0 := \{(x_0, \alpha_0)\}$
Selection of Poll center: Order the filter $\mathcal{F}_k$ in some way and select a poll center.
Poll step: Choose $D_k \subseteq \mathcal{D}$. Evaluate F_X and h at the set of poll points $P_k := \{x_k + \alpha_k d \mid d \in D_k\}$.
$\mathcal{F}_{k+1} = \mathcal{F}_k$?
NO
YES
$\alpha_{k+1} \in [\alpha_k, \gamma\alpha_k]$
$\alpha_{k+1} \in [\beta_1\alpha_k, \beta_2\alpha_k]$

DMS-Filter



Solutions: $L := \{(x, \alpha) \in \mathcal{F} \mid (F_X(x), h(x)) = (F(x), 0)\}$.

DMS-Filter – Example



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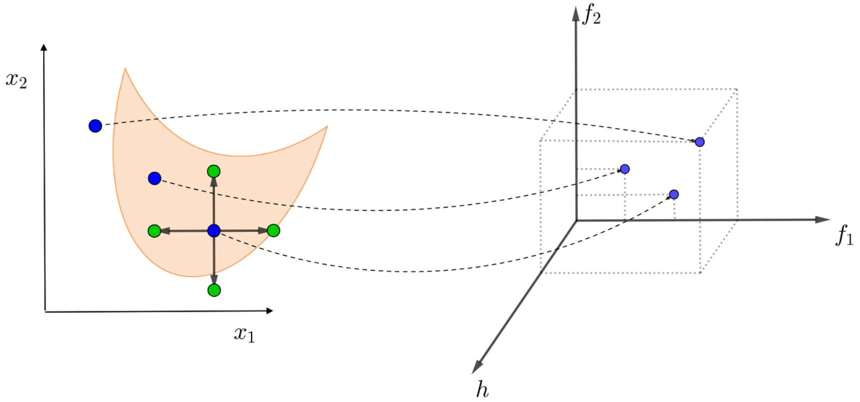

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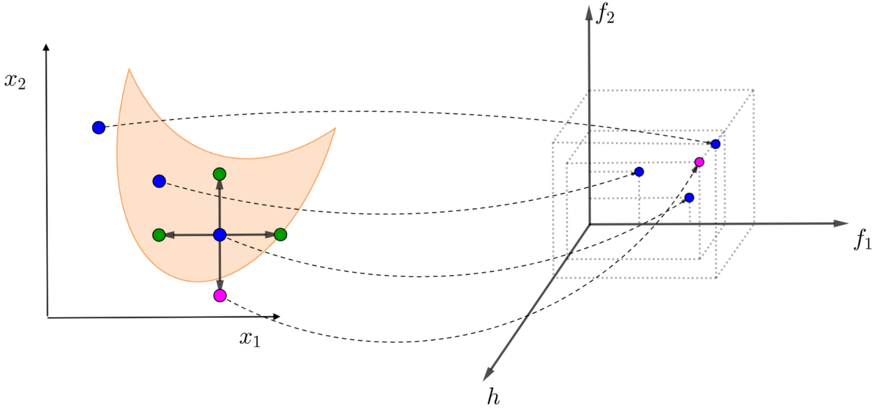

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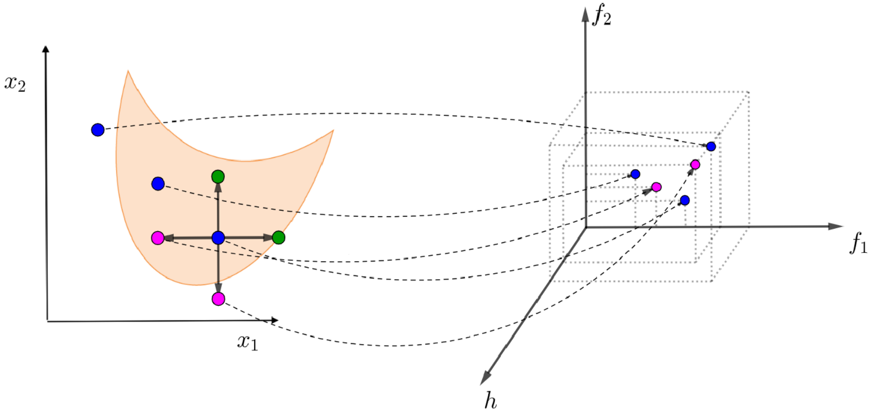

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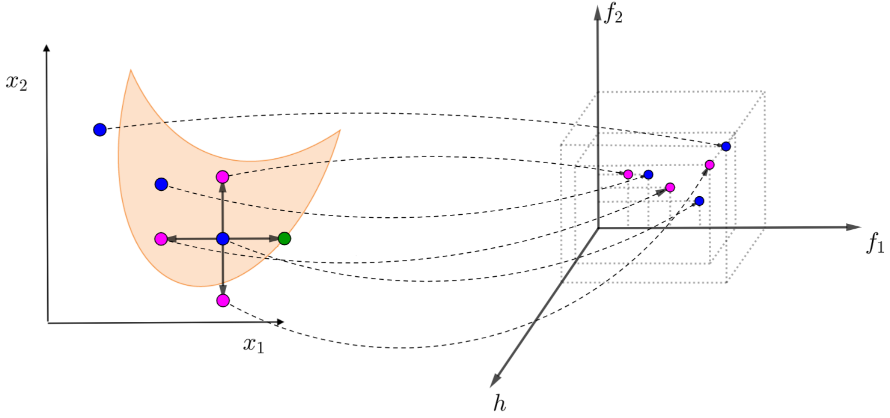

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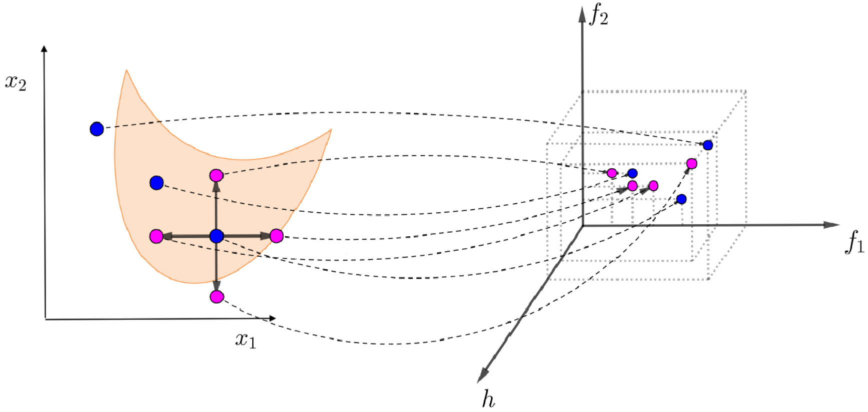

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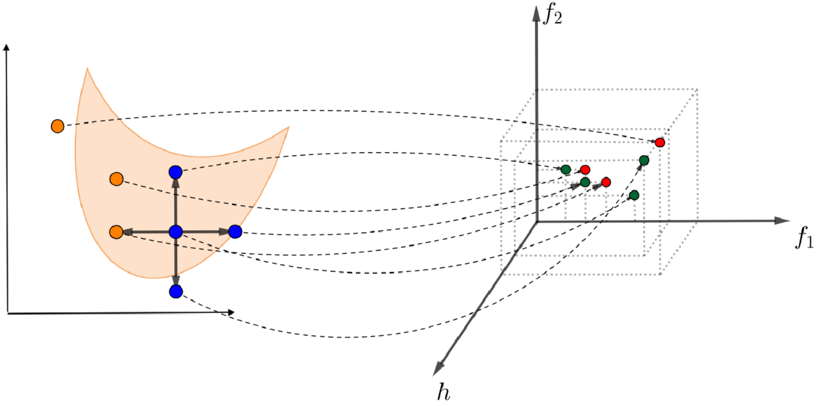

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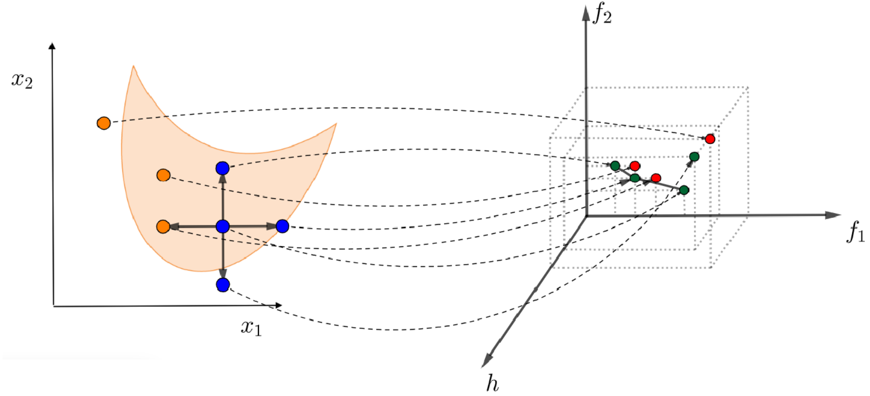
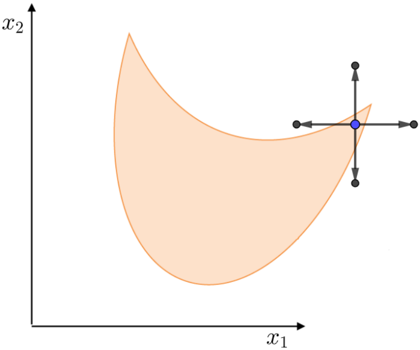
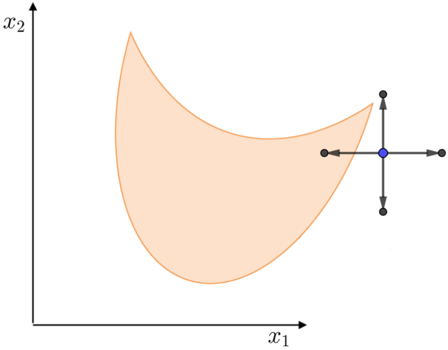


Poll Center Change

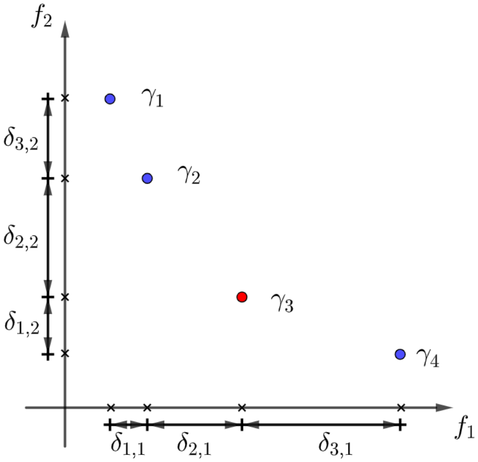
- **Feasible to Infeasible**



- **Infeasible to Feasible**



Selection of the most isolated point



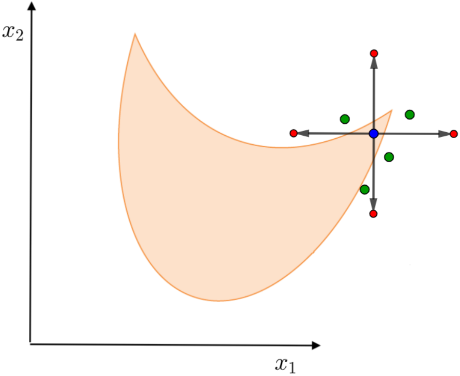
$$\gamma_1 = \frac{\delta_{3,2} + \delta_{1,1}}{2}$$

$$\gamma_2 = \frac{\frac{\delta_{3,2}+\delta_{2,2}}{2} + \frac{\delta_{1,1}+\delta_{2,1}}{2}}{2}$$

$$\gamma_3 = \frac{\frac{\delta_{2,2}+\delta_{1,2}}{2} + \frac{\delta_{2,1}+\delta_{3,1}}{2}}{2}$$

$$\gamma_4 = \frac{\delta_{1,2} + \delta_{3,1}}{2}$$

Infeasible poll center



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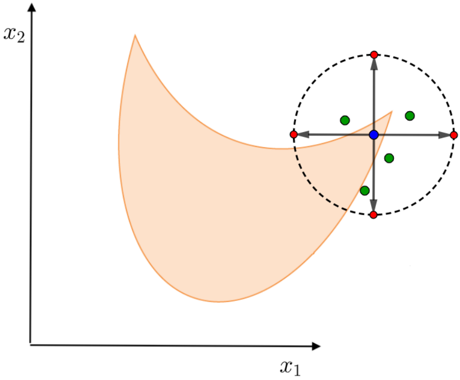

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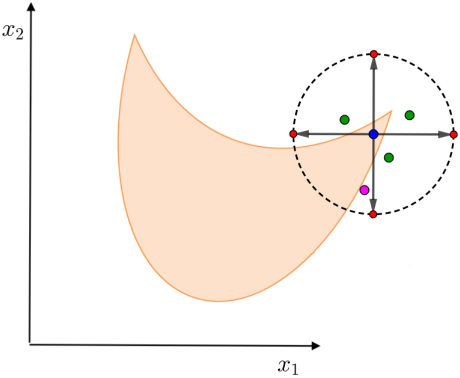

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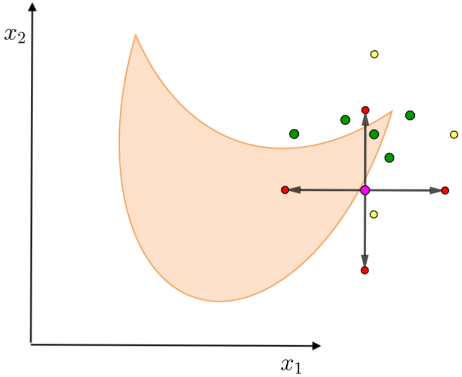
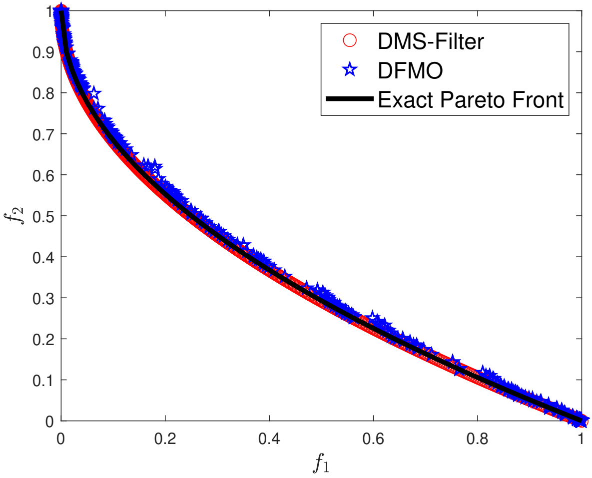


L1ZDT4 constrained Problem – DMS-Filter versus DFMO



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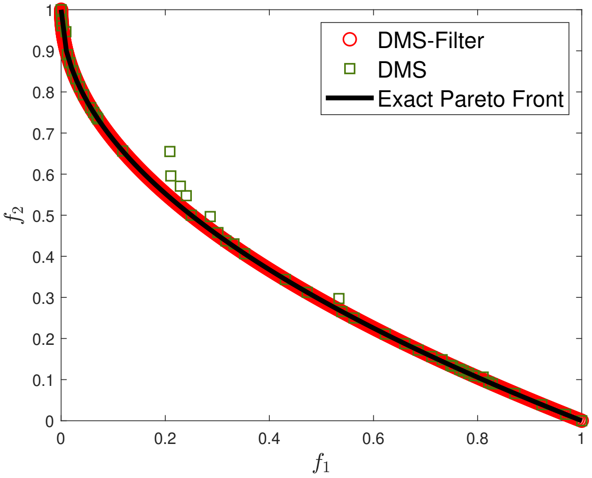

Convergence Results

Theorem (Refining Subsequences)

There is at least a convergent refining subsequence of iterates $\{x_k\}_{k \in K}$, corresponding to unsuccessful poll steps, such that $\lim_{k \in K} \alpha_k = 0$.

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Let $\bar{x}$ be the limit point of a convergent refining subsequence $\{x_k\}_{k\in K}$.

Definition (Refining Directions)

Refining directions for $\bar{x}$ are limit points of $\{d_k/\|d_k\|\}_{k\in K}$, where $d_k \in D_k$ and $x_k + \alpha_k d_k \in \mathcal{S} := S \cup \{x \in X \mid h(x) \leq h_{\mathtt{max}}\}$

Convergence Results

Consider a refining subsequence converging to $\overline{x}$ (and assume that F and h are Lipschitz continuous near $\overline{x}$)

Theorem

If $d \in \text{int}(T_X^{Cl}(\overline{x}))$ is a refining direction for $\overline{x}$ then:

$$\exists j = j(d) \in \{1, \ldots, p\} \; : \; f_j^\circ(\overline{x}; d) \geq 0 \quad \text{or} \quad h^\circ(\overline{x}; d) \geq 0$$

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Theorem

If the set of refining directions for $\overline{x}$ is dense in $\operatorname{int}(T_X^{Cl}(\overline{x})) \neq \emptyset$ then

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Conclusions and Future Work

- DMS-Filter presents competitive numerical results for constrained Bi-objective Derivative-free Optimization Problems

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- DMS-Filter presents competitive numerical results for constrained Bi-objective Derivative-free Optimization Problems
- Extend the approach to problems with more than two objectives

THANKS FOR YOUR ATTENTION!

Any comments or questions?

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