Addressing Constraints in Direct Multisearch

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UI/BD/151246/2021 UIDB/00297/2020 UIDP/00297/2020

Presentation Outline

1. Problem Definition
2. DMS
3. DMS-FILTER-IR
4. LOG-DMS
5. Conclusions and Future Work

Presentation Outline

Problem Definition

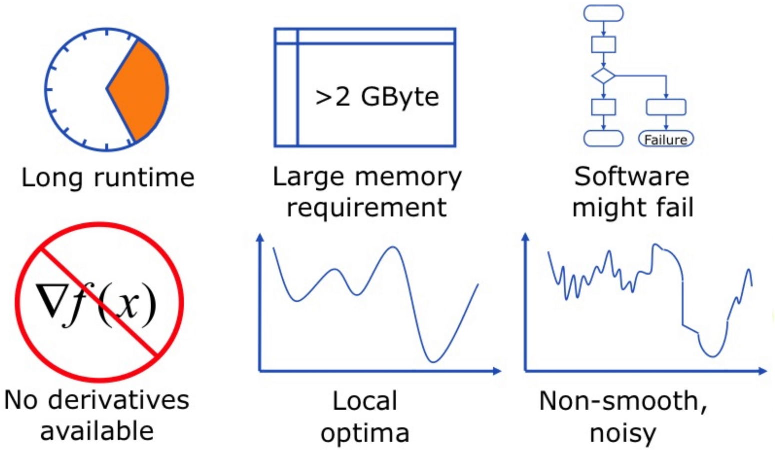
Multiobjective Derivative-free Optimization

$$\min_{x \in \Upsilon \subseteq \mathbb{R}^n} F(x) \equiv (f_1(x), f_2(x), \ldots, f_m(x))^\top$$

$$f_j : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, \; j = 1, \ldots, m \geq 2$$

- several objectives, often conflicting
- expensive function evaluation
- impossible to use or approximate derivatives

Blackboxes as illustrated by J. Simonis [ISMP 2009]



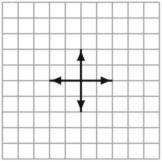
Iterative Methods

$$x_{k+1} = x_k + \alpha_k d_k$$

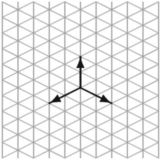
- Derivative-based methods: d_k should be a descent direction according to at least one of the objectives, i.e.

$$d_k^\top \nabla f_i(x_k) < 0, \quad \text{with } i \in \{1, \ldots, m\}$$

- Derivative-free methods: when derivatives are not available and cannot be numerically approximated
  - Directional Direct Search: Uses positive spanning sets for sampling
    In $\mathbb{R}^2$:



$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$



$$\begin{bmatrix} \sqrt{3} & 0 & -\sqrt{3} \\ -1 & 2 & -1 \end{bmatrix}$$

$pspan(D) = \mathbb{R}^2$

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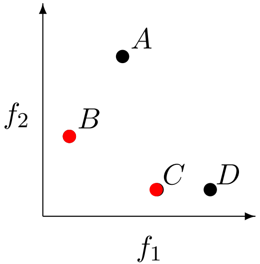
Direct MultiSearch (DMS) Main Lines

- does not aggregate any of the objective function components
- makes use of Pareto dominance

Pareto Dominance (x dominates y)

$F(x) \leq F(y)$, with $F(x) \neq F(y)$



Pareto Front: $\{B, C\}$

- generalizes directional direct search to MOO
- considers the search/poll paradigm with an optional search step
- computes approximations to the complete Pareto front

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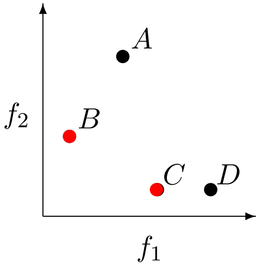

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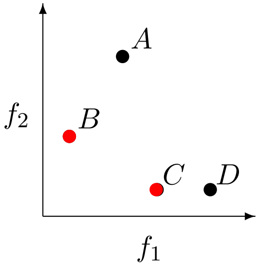

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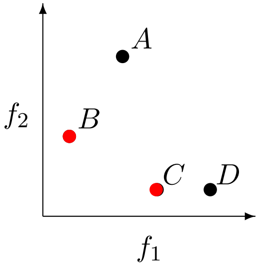

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Direct MultiSearch (DMS) Main Lines

- constraints are addressed by an extreme barrier approach

$$F_{\Upsilon}(x) = \begin{cases} F(x) & \text{if } x \in \Upsilon, \\ (+\infty, +\infty, \ldots, +\infty)^{\top} & \text{otherwise} \end{cases}$$

- keeps a list of feasible nondominated points
- poll centers are chosen from the list
- successful iterations correspond to list changes

successful iteration $\Leftrightarrow$ new feasible nondominated point

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Direct Multisearch Structure

Initialization

Choose $x_0 \in \Upsilon$ with $F(x_0) < +\infty$, $\alpha_0 > 0$. Set $L_0 = \{(x_0; \alpha_0)\}$

Selection of iterate point

Order L_k and select $(x_k; \alpha_k) \in L_k$

Search Step (Optional)

Evaluate a finite set of points $L_{add} = \{(z_s; \alpha_k)\}_{s \in S}$

$(L_k; L_{add}) \hookrightarrow L_{filtered} \hookrightarrow L_{trial}$

Suc: $L_{k+1} = L_{trial}$

Unsuc

Poll Step

Evaluate $L_{add} = \{(x_k + \alpha_k d; \alpha_k),\ d \in D_k\}$, with $D_k \subseteq \mathcal{D}$

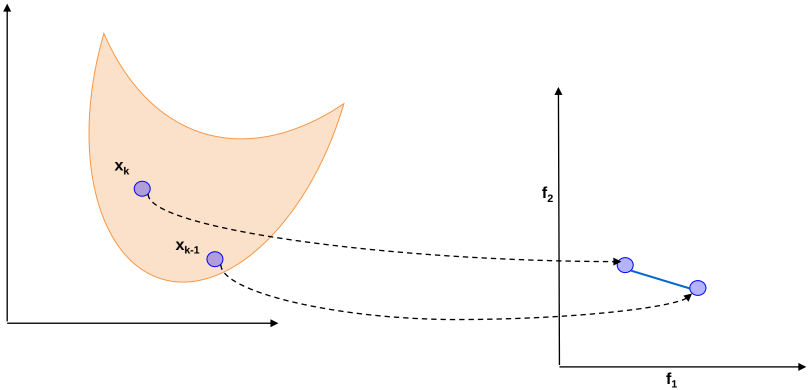
$(L_k; L_{add}) \hookrightarrow L_{filtered} \hookrightarrow L_{trial}$

Suc: $L_{k+1} = L_{trial}$

Unsuc

Decrease the stepsize

Poll Step Example (Biobjective Problem)



L_k

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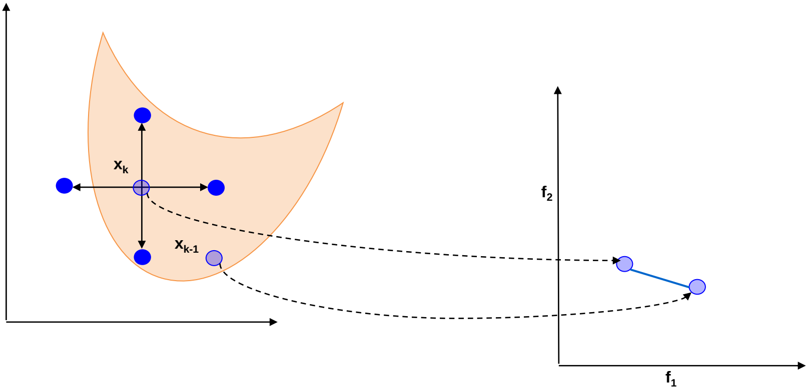

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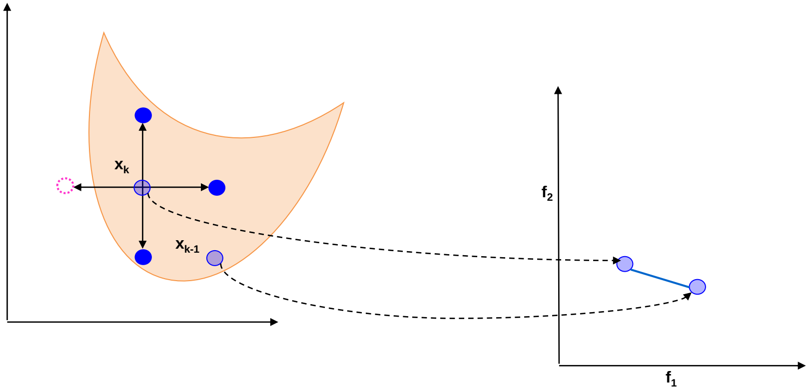

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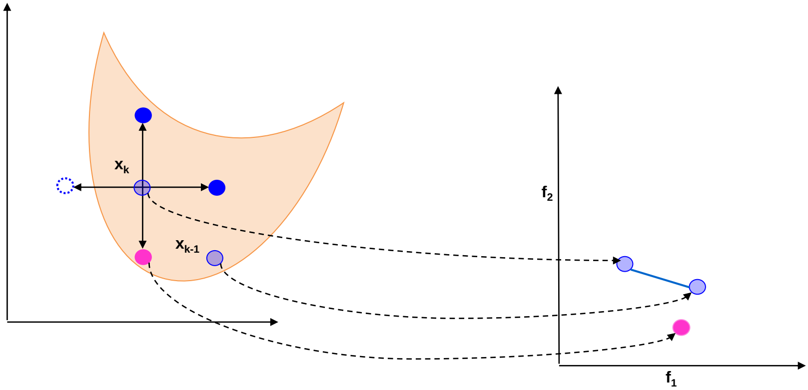

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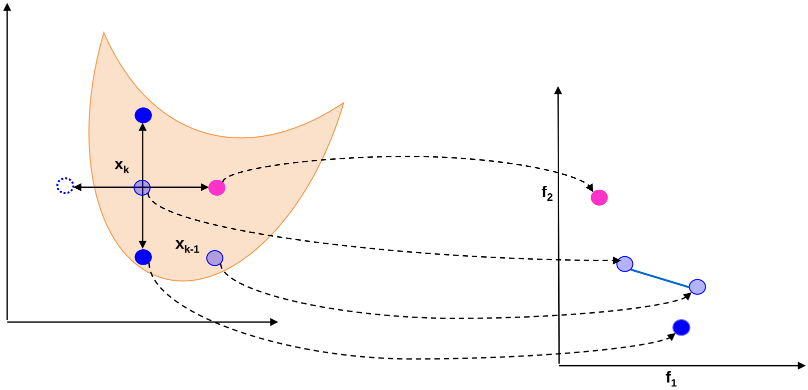

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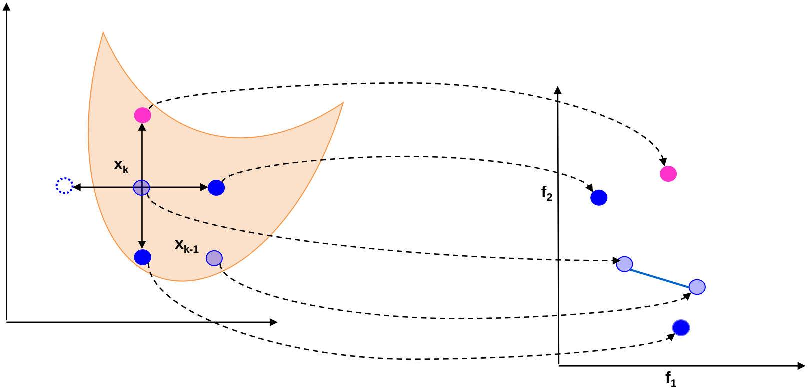

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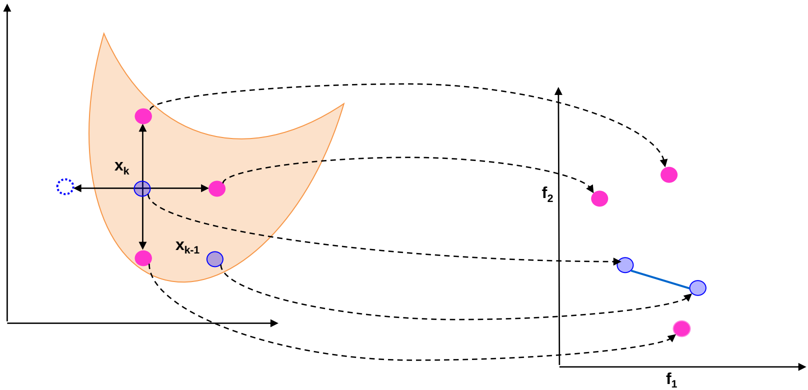
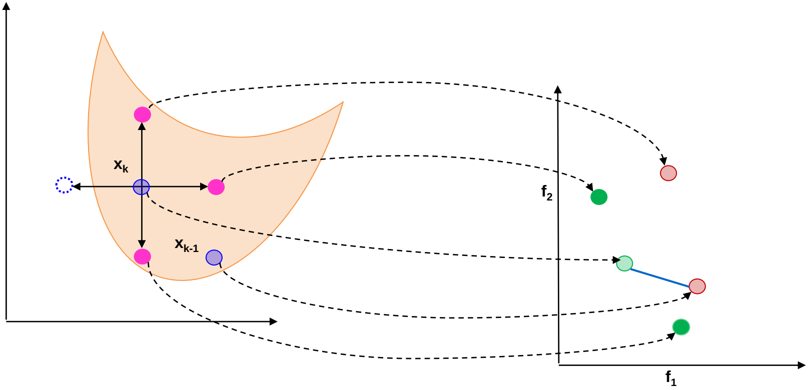


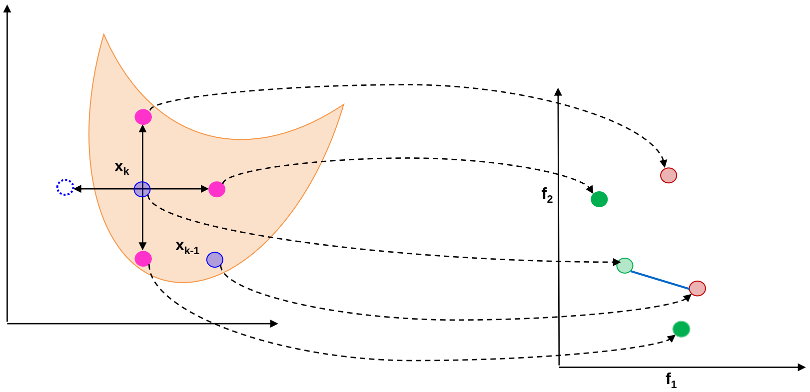
L_{add}

Poll Step Example (Biobjective Problem)



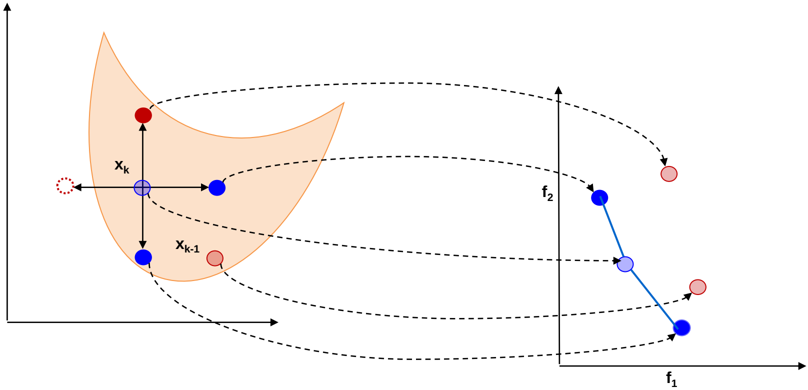
$L_{filtered}$

Poll Step Example (Biobjective Problem)



$$L_{trial} = L_{filtered}$$

Poll Step Example (Biobjective Problem)



L_{k+1}

Positive Spanning Sets and Constraints

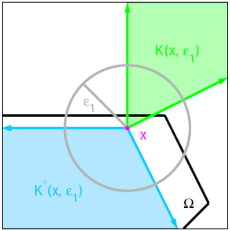
- Requires asymptotical density of the union of positive spanning sets in the unit sphere

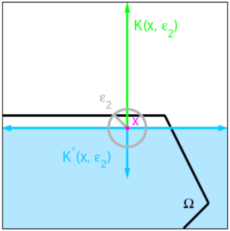
In practice:

- Unconstrained or variable bounds: coordinate search

$$D = D_{\oplus} = [\, I \;\; -I \,]$$

- Linear constraints: directions must conform to geometry of nearby constraints





Abramson, Brezhneva, Dennis, and Pingel [2008]

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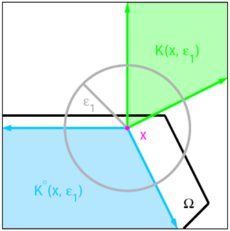

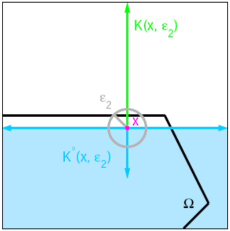

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Metrics for Performance Profiles (Dolan and Moré [2002])

- Purity

$$\frac{|F_{p,s} \cap F_p|}{|F_{p,s}|}$$

- Spreads Γ and Δ

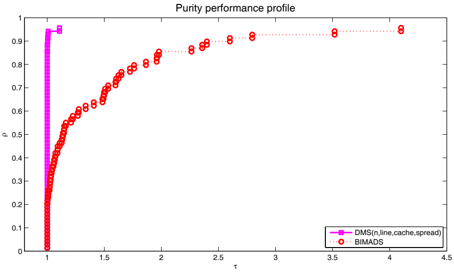
$$\Gamma_{p,s} = \max_{j \in \{1,\ldots,m\}} \left(\max_{i \in \{0,\ldots,N\}} \{d_i\} \right)$$

$$\Delta = \max_{j \in \{1,\ldots,m\}} \left(\frac{d_0 + d_N + \sum_{i=1}^{N-1} |d_i - \overline{d}|}{d_0 + d_N + (N-1)\overline{d}} \right)$$

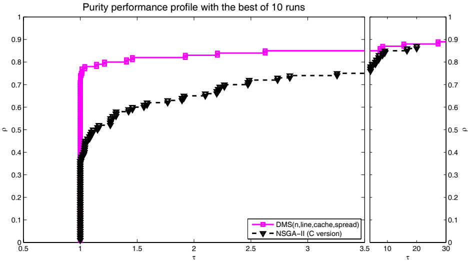
DMS Defaults

- List initialization: line sampling
- Search step: none
- Poll center selection: poll centers correspond to highest Γ metric value (ties broken by the largest step size)
- Polling strategy: complete coordinate search
- Stepsize update: halved at unsuccessful iterations
- Cache implementation: objective function values only computed for points that dist at least 10^{-3} from any previously evaluated point

Comparing DMS with Other Solvers (Purity)



Purity Metric
(percentage of points generated in the reference Pareto front)

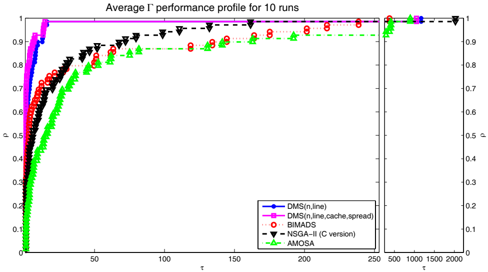


DMS
BIMADS
NSGA-II

Comparing DMS with Other Solvers (Spread)

Gamma Metric

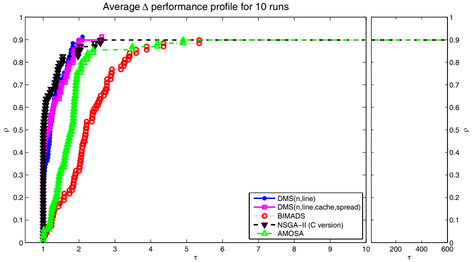
(largest gap in the Pareto front)



Delta Metric

(uniformity of gaps in the Pareto front)

DMS, **DMS**, **BIMADS**, **NSGA-II**, **AMOSA**



Motivation

- DMS $\rightarrow$ A. L. Custódio, J. F. A. Madeira, A. I. F. Vaz, and L. N. Vicente, *Direct multisearch for multiobjective optimization*, SIAM J. Optim. (2011), 21, 1109–1140

- DFMO $\rightarrow$ G. Liuzzi, S. Lucidi, and F. Rinaldi, *A derivative-free approach to constrained multiobjective nonsmooth optimization*, SIAM J. Optim. (2016), 26, 2744–2774

- 93 biobjective problems with nonlinear constraints and variable bounds
  - number of variables between 3 and 30
  - number of nonlinear constraints between 1 and 29
  - maximum of 20000 function evaluations

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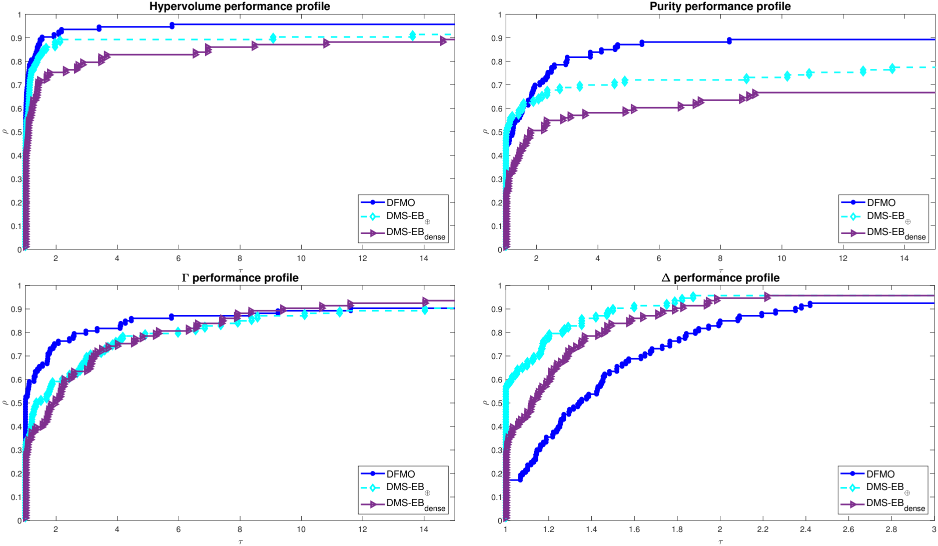
- Hypervolume

$$HI_{p,s} = Vol\{b \in \mathbb{R}^m \mid b \leq U_p \wedge \exists a \in F_{p,s} : a \leq b\}$$

Nonlinear + Bound Constraints (Biobjective Problems)



DFMO **DMS-EB-Coordinate** **DMS-EB-Dense**

Presentation Outline

DMS Filter and Inexact Restoration Approach

Original Problem

$$\min_{x \in \Upsilon \subset \mathbb{R}^n} \quad F(x) = (f_1(x), f_2(x), \ldots, f_m(x))^\top$$

$$f_j : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, \; j = 1, 2, \ldots, m \geq 2$$

with $\Upsilon = \Omega \cap X$ (where: Ω relaxable and X unrelaxable)

Reformulated Problem

$$\min_{x \in X} \; (f_1(x), f_2(x), \ldots, f_m(x), h(x))^\top$$

Constraint Violation function

$$h(x) = \|C(x)_+\|_2^2 = \sum_{i=1}^{p} \max\{0, c_i(x)\}^2$$

Constraints in X continue to be addressed by an extreme barrier approach, and it is assumed $x_0 \in X$.

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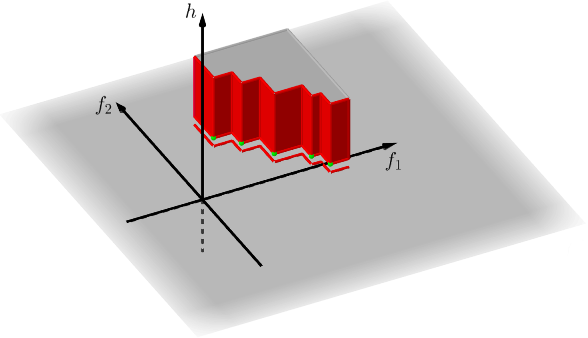
List of Nondominated Points

A point x' is said to be filtered by the list L if any of the following properties hold:

- $h(x') > h_{\texttt{max}}$ (for some fixed $h_{\texttt{max}} > 0$)
- there is $x \in L$ such that $(F(x), h(x)) \leq (F(x'), h(x'))$ with $(F(x), h(x)) \neq (F(x'), h(x'))$



DMS Filter and Inexact Restoration Approach

- Relaxable feasibility is treated as an additional objective

- Priority given to feasible poll centers

- When all poll points associated with a poll center x_k are infeasible, switches to an infeasible poll center

  Attempts to restore feasibility by solving:

$$\begin{aligned} \min_{y \in X} \quad & \frac{1}{2}\|y - x_k\|^2 \\ \text{s.t.} \quad & h(y) \leq \xi(\alpha_k) h(x_k), \end{aligned}$$

where $\xi : (0, +\infty) \to (0, 1)$, is continuous, and satisfies

$$\xi(t) \to 0 \text{ when } t \downarrow 0$$

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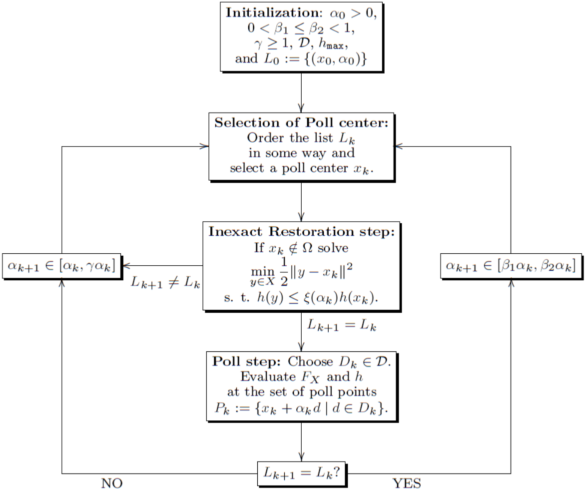
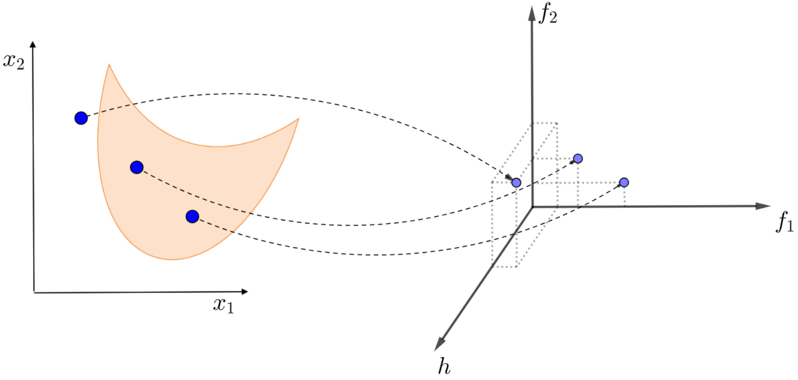
DMS-FILTER-IR – Algorithmic Structure



Solutions$:= \{(x, \alpha) \in L \mid (F(x), h(x)) = (F(x), 0)\}$

DMS-FILTER-IR – Poll Step



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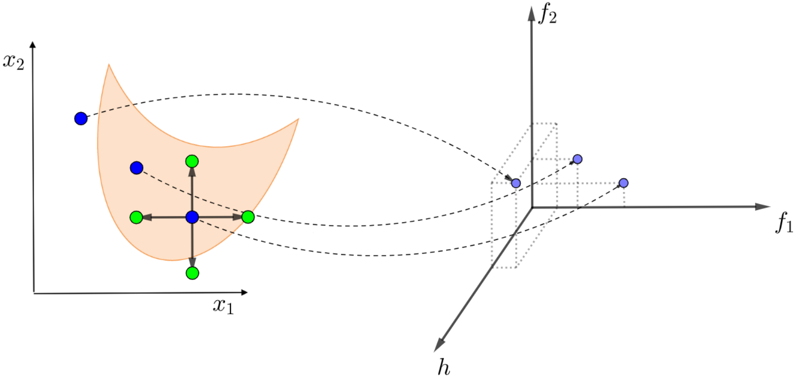

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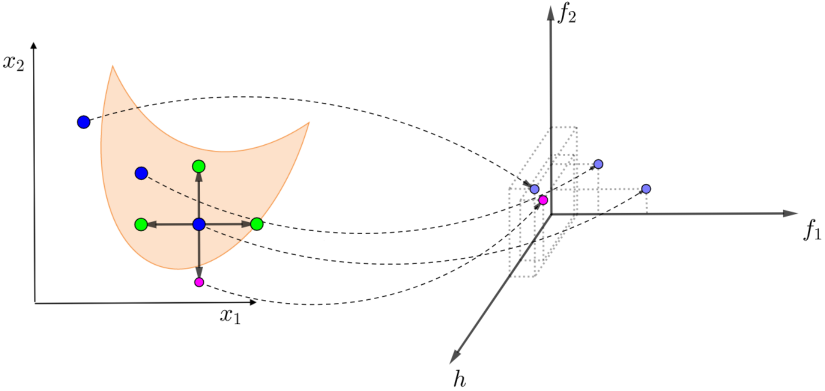

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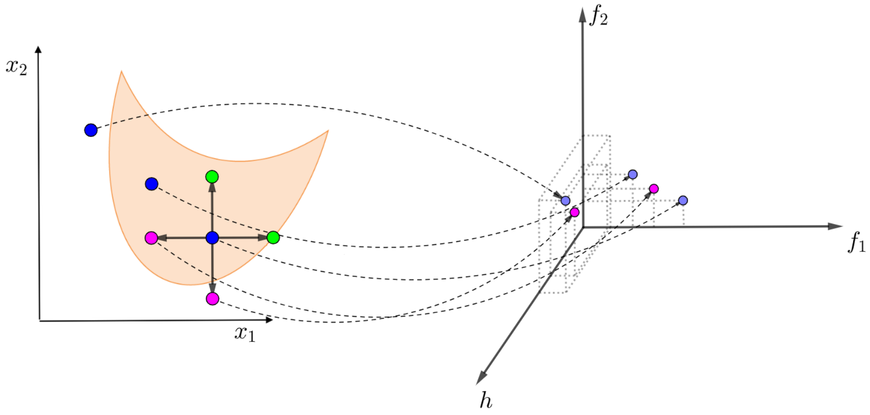

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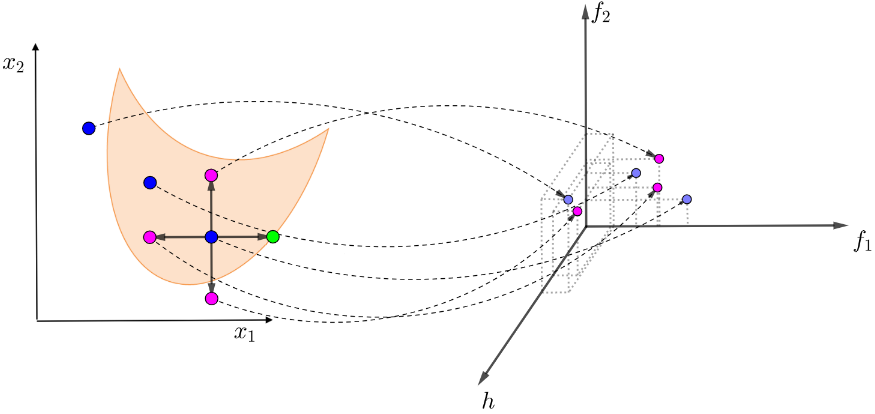

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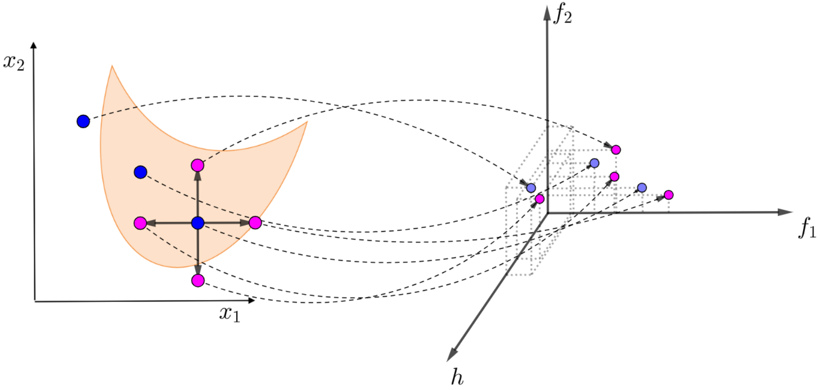

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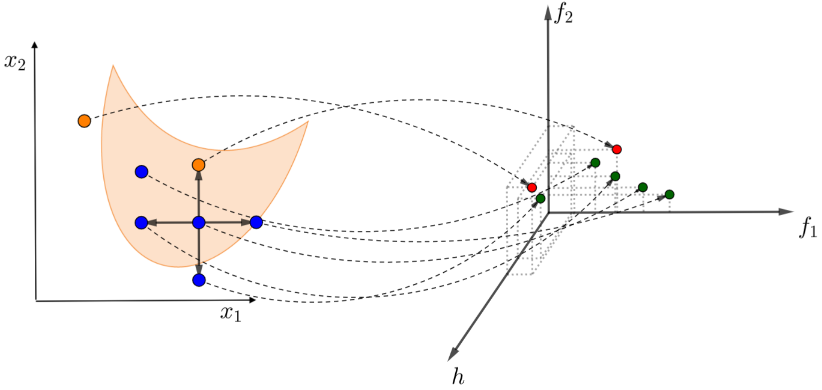
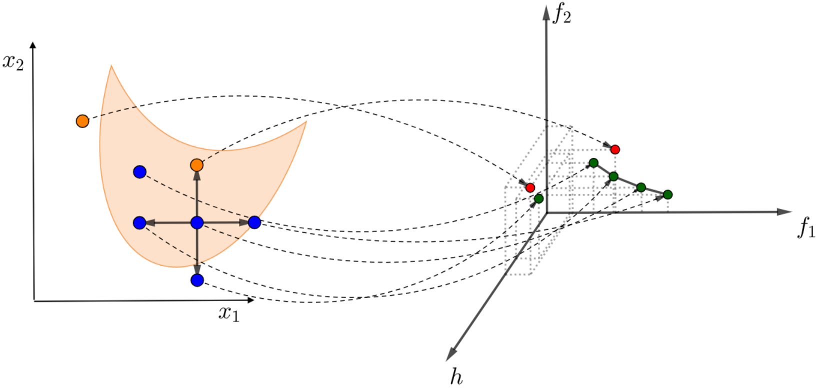


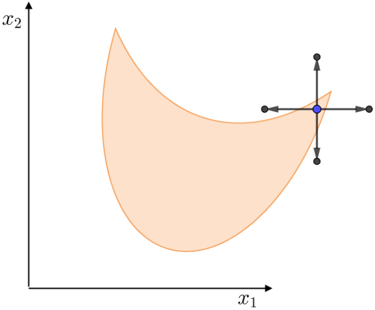
DMS-FILTER-IR – Poll Step



Poll Center Selection

Feasible to Infeasible

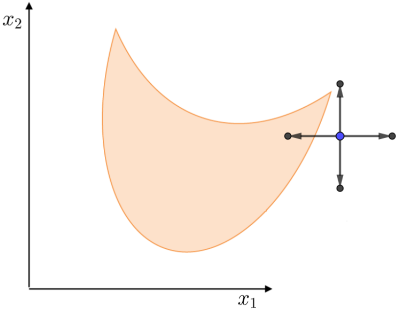
Polling only generates infeasible points



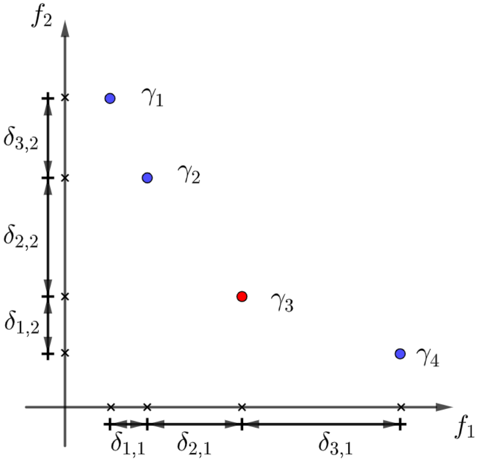
Infeasible to Feasible

Infeasible x_k generates feasible point by inexact restoration or polling

$$\begin{aligned} \min_{y \in X} \quad & \frac{1}{2}\|y - x_k\|^2 \\ \text{s.t.} \quad & h(y) \leq \xi(\alpha_k) h(x_k) \end{aligned}$$



Feasible poll center - Most Isolated Point



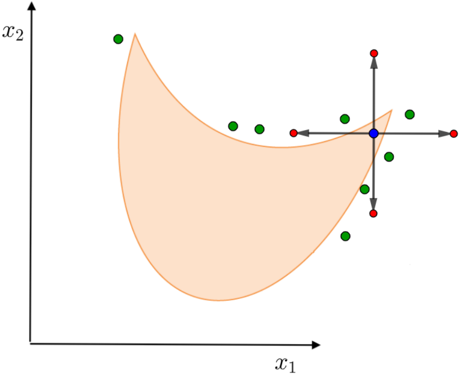
$$\gamma_1 = \frac{\delta_{3,2} + \delta_{1,1}}{2}$$

$$\gamma_2 = \frac{\frac{\delta_{3,2}+\delta_{2,2}}{2} + \frac{\delta_{1,1}+\delta_{2,1}}{2}}{2}$$

$$\gamma_3 = \frac{\frac{\delta_{2,2}+\delta_{1,2}}{2} + \frac{\delta_{2,1}+\delta_{3,1}}{2}}{2}$$

$$\gamma_4 = \frac{\delta_{1,2} + \delta_{3,1}}{2}$$

Infeasible Poll Center



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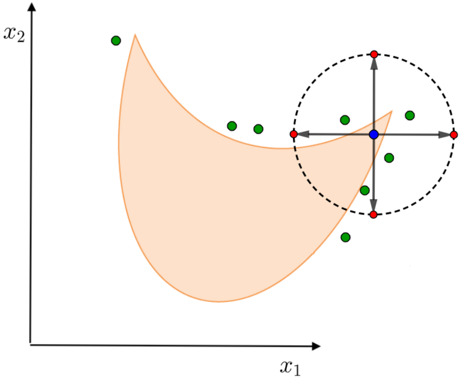

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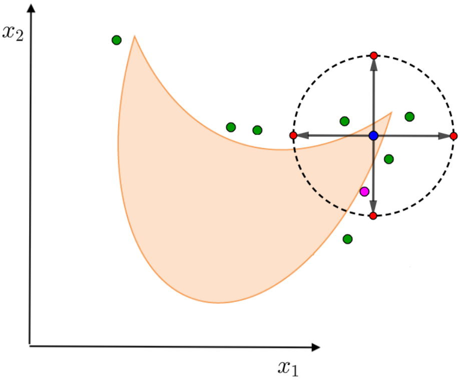

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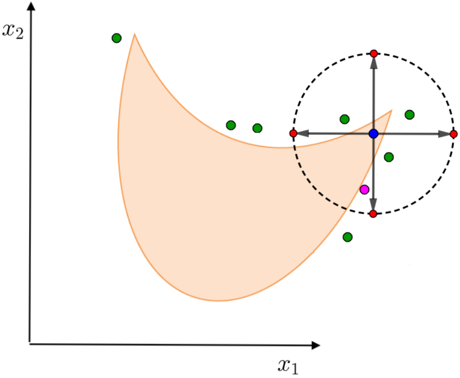
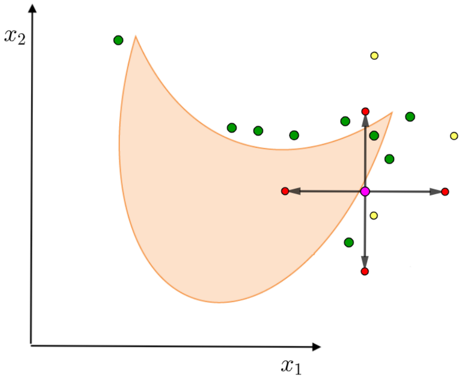


$$\min_{y \in X} \quad \frac{1}{2}\|y - x_k\|^2$$
$$\text{s.t.} \quad h(y) \leq \xi(\alpha_k) h(x_k)$$

Infeasible Poll Center



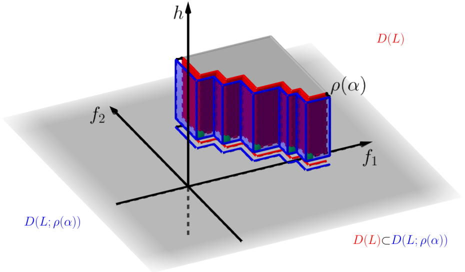
Convergence Analysis – Globalization Strategies

Using Integer Lattices (Torczon [1997], Audet and Dennis [2002])

- requires only simple decrease
- poll directions and step size must satisfy integer/rational requirements

Imposing Sufficient Decrease (Kolda, Lewis, and Torczon [2003])

- use of a forcing function $\rho : (0, +\infty) \to (0, +\infty)$, continuous, nondecreasing, and satisfying $\rho(t)/t \to 0$ when $t \downarrow 0$
- x is nondominated $\Leftrightarrow (F_X(x), h(x)) \notin D(L, \rho(\alpha))$



Convergence Results – Sequences

Refining sequence

A sequence $\{x_k\}_{k\in K}$, such that $k \in K$ is an unsuccessful iteration and $\lim_{k\in K} \alpha_k = 0$.

Theorem (Refining Subsequences)

There is at least a convergent refining subsequence of iterates $\{x_k\}_{k\in K}$, corresponding to unsuccessful poll steps, with $\lim_{k\in K} \alpha_k = 0$.

Let $\overline{x}$ be the limit point of a convergent refining subsequence $\{x_k\}_{k\in K}$.

Definition (Refining Directions)

Refining directions for $\overline{x}$ are limit points of $\{d_k/\|d_k\|\}_{k\in K}$, where $d_k \in D_k$ and $x_k + \alpha_k d_k \in \mathcal{S} := \{x \in X \mid h(x) \leq h_{\mathtt{max}}\}$.

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Convergence Results

Consider $\{x_k\}_{k\in K}$ a refining subsequence converging to $\overline{x} \in \mathcal{S} := \{x \in X \mid h(x) \leq h_{\texttt{max}}\}$. Assume that F and h are Lipschitz continuous near $\overline{x}$. Under any globalization strategy:

Theorem

- If $d \in \operatorname{int}(T_{\mathcal{S}}^{Cl}(\overline{x}))$ is a refining direction for $\overline{x}$ then:

$$\exists j = j(d) \in \{1, \ldots, m+1\} \text{ such that } \quad f_j^{\circ}(\overline{x}; d) \geq 0$$

- If the set of refining directions for $\overline{x}$ is dense in $\operatorname{int}(T_{\mathcal{S}}^{Cl}(\overline{x})) \neq \emptyset$ then $\overline{x}$ is a Pareto-Clarke critical point of $\overline{F}$ in $\mathcal{S}$:

$$\forall d \in T_{\mathcal{S}}^{Cl}(\overline{x}), \ \exists j = j(d) \in \{1, \ldots, m+1\} \ \text{ such that } \quad f_j^{\circ}(\overline{x}; d) \geq 0$$

Convergence Results – Infeasible case

Theorem

Let h be continuous and consider $\{x_k\}_{k \in K}$ an **infeasible refining subsequence** such that for each $k \in K$, x_k is **used at a successful inexact restoration step**. Then DMS-FILTER-IR generates a **limit point** $\overline{y} \in \Upsilon$.

Convergence Results – Feasible case

Consider $\{x_k\}_{k\in K}$ a **feasible refining subsequence** converging to $\overline{x} \in \Upsilon$. Assuming a **globalization strategy based on integer lattices**, we have:

Corollary

- If $d \in \mathrm{int}(T_{\Upsilon}^{Cl}(\overline{x}))$ is a refining direction for $\overline{x}$ then:

$$\exists j = j(d) \in \{1,\dots,m\} \text{ such that } \; f_j^{\circ}(\overline{x}; d) \geq 0$$

- If the set of refining directions for $\overline{x}$ is dense in $\mathrm{int}(T_{\Upsilon}^{Cl}(\overline{x})) \neq \emptyset$ then $\overline{x}$ is a Pareto-Clarke critical point of F in Υ:

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Numerical Results – Settings

- Comparison between:
  - **DMS-EB** versus **DMS-FILTER-IR**: best version of each one
- 93 biobjective problems with nonlinear constraints and bounds
  - number of variables between 3 and 30
  - number of nonlinear constraints between 1 and 29
- Initialization:
  - **DMS-EB**: Feasible point provided by Karmitsa [2007]
  - **DMS-FILTER-IR**: n-points equally spaced in the line segment, joining the variable upper and lower bounds
- Stopping criterion
  - $\alpha_k < 10^{-3}$ for all points in the list
  - Maximum of 5000 function evaluations

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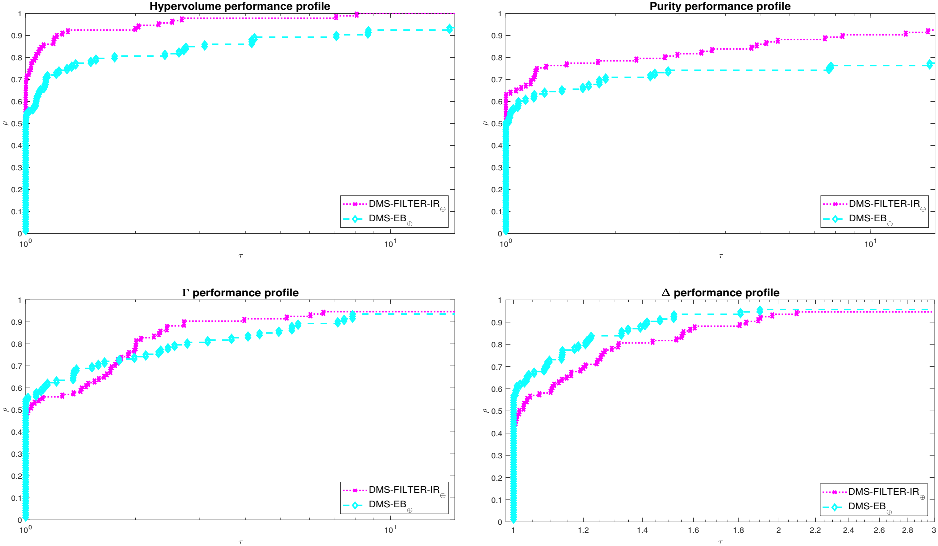
Best version DMS vs DMS-FILTER-IR - 5k func. eval.



DMS-FILTER-IR - Coordinate **DMS-EB - Coordinate**

Numerical Settings

- Comparison among **DFMO**, **DMS-EB**, **DmultiMads-PB*** and **DMS-FILTER-IR**
- 93 biobjective problems with nonlinear constraints and bounds
  - number of variables between 3 and 30
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  - **Others:** n-points equally spaced in the line segment, joining the variable upper and lower bounds
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  - **DFMO** and **DmultiMads-PB**:
    - default values
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* **DmultiMads-PB** → Jean Bigeon, Sébastien Le Digabel, Ludovic Salomon. *Handling of constraints in multiobjective blackbox optimization*. COAP (2024).

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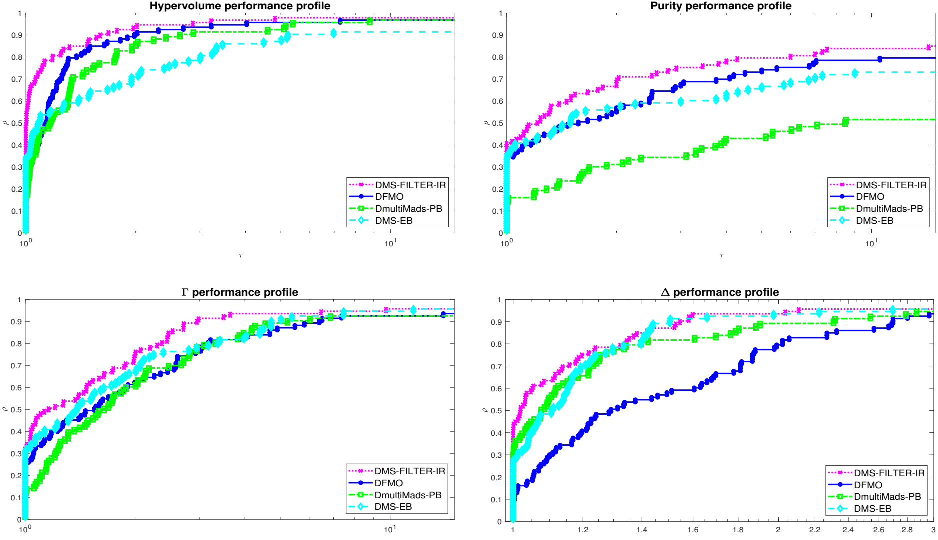
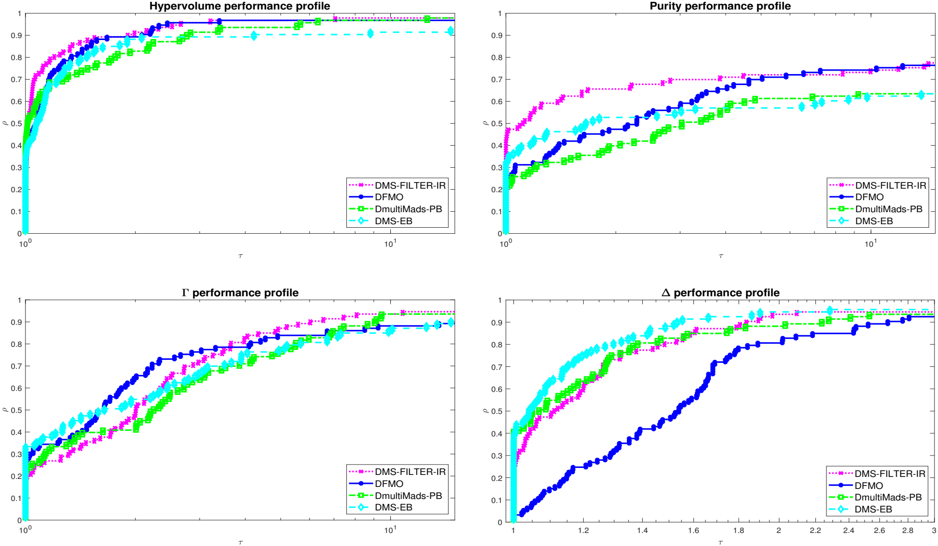
Results - 500 function evaluations



DMS-FILTER-IR **DFMO** **DmultiMads-PB** **DMS-EB**

Results - 5000 function evaluations



DMS-FILTER-IR DFMO DmultiMads-PB DMS-EB

Presentation Outline

1. Problem Definition
2. DMS
3. DMS-FILTER-IR
4. LOG-DMS
5. Conclusions and Future Work

Logarithmic Barrier in Direct Multisearch

Original Problem

$$\begin{aligned} \min \;& F(x) = (f_1(x), \ldots, f_m(x))^\top \\ \text{s.t.} \;& g(x) \le 0 \\ & x \in X \end{aligned}$$

$$F : X \subseteq \mathbb{R}^n \to \{\mathbb{R} \cup \{+\infty\}\}^m$$
$$g : X \subseteq \mathbb{R}^n \to \{\mathbb{R} \cup \{+\infty\}\}^p$$
$$X := \{x \in \mathbb{R}^n \mid lb \le x \le ub\}$$

Logarithmic Barrier function

$$Z_\ell(x;\rho) = \begin{cases} f_\ell(x) - \rho \sum_{i=1}^{p} \log\big(-g_i(x)\big) & \text{if } x \in X \text{ and } \max_{i=1,\ldots,p} g_i(x) < 0, \\ +\infty & \text{otherwise.} \end{cases}$$

where $\ell = 1, \ldots, m$ and $\rho > 0$.

- ρ must be driven to zero

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where $\ell = 1, \ldots, m$ and $\rho > 0$.

- ρ must be driven to zero

Initialization

Choose $x_0 \in X$ such that $g(x_0) < 0$, $\alpha_0 > 0$, $\mathcal{D}$, $\rho_0 > 0$, $\eta \geq 1$, $\theta \in (0,1)$. Set $L_0 = \{(x_0; \alpha_0)\}$

Selection of iterate point

Order L_k and select $(x_k; \alpha_k) \in L_k$

Search Step (Optional)

Evaluate a finite set of points $L_{add} = \{(z_s; \alpha_k)\}_{s \in S}$

$(L_k; L_{add}) \hookrightarrow L_{filtered} \hookrightarrow L_{trial}$

Suc: $L_{k+1} = L_{trial}$

Unsuc

Poll Step

Evaluate $L_{add} = \{(x_k + \alpha_k d; \alpha_k),\ d \in D_k\}$, with $D_k \subseteq \mathcal{D}$

$(L_k; L_{add}) \hookrightarrow L_{filtered} \hookrightarrow L_{trial}$

Suc: $L_{k+1} = L_{trial}$

Unsuc

Decrease the stepsize

$\alpha_{k+1} > \min\{\rho_k^{\beta}, (g_{\min})_k^2\}$

$\alpha_{k+1} \leq \min\{\rho_k^{\eta}, (g_{\min})_k^2\}$

Penalty Update

$\rho_{k+1} = \theta \rho_k$

$$(g_{\min})_k = \min_{x \in L_k} \left\{ \min_{i=1,\ldots,p} \{|g_i(x_k)|\} \right\}$$

Numerical Settings

- Comparison among **DMS**, **LOG-DMS**, and **DMS-FILTER-IR**
- Stopping criterion
  - **DMS**, **LOG-DMS**, and **DMS-FILTER-IR**:
    - $\alpha_k < 10^{-3}$ for all points in the list
    - **All:** maximum of 5000 function evaluations
- **LOG-DMS** maintains a cache
  - $P^{\text{cache}} \in \mathbb{R}^{n \times |\text{cache}|}$ - the points
  - $F^{\text{cache}} \in \mathbb{R}^{m \times |\text{cache}|}$ - values of the objective function
  - $G^{\text{cache}} \in \mathbb{R}^{p \times |\text{cache}|}$ - values of the constraints
- If the ρ_k is updated $\rightarrow$ recompute $Z(\cdot; \rho_{k+1})$
- $\eta = 1$, $\theta = 10^{-2}$, and $\rho = 1$
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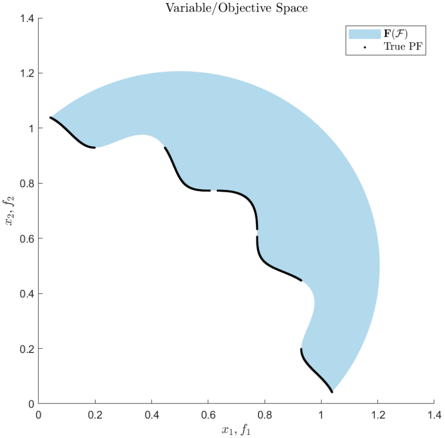
$$\min \; F(x) = (x_1, x_2)^\top$$

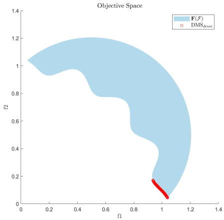
$$s.t. \;\; g_1(x) = -x_1^2 - x_2^2 + 1 + 0.1\cos\left(16\arctan\left(\frac{x_1}{x_2}\right)\right) \leq 0$$

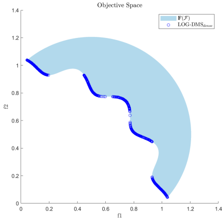
$$g_2(x) = (x_1 - 0.5)^2 + (x_2 - 0.5)^2 - 0.5 \leq 0$$

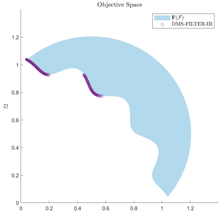
$$x \in X = [0, \pi] \times [0, \pi]$$

$$x_0 = (1.1, 0.15)^\top$$









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Conclusions

- DMS-FILTER-IR extends filter methods, with an inexact restoration step, to the DMS framework
- DMS-FILTER-IR presents a well-supported convergence analysis
- DMS-FILTER-IR presents competitive numerical results for constrained biobjective derivative-free optimization problems
- LOG-DMS keeps the basic algorithmic features of DMS
- LOG-DMS provides a log-barrier approach to address nonlinear constraints in DMS

Future Work

- developing a competitive numerical implementation for problems with more than two objectives for DMS-FILTER-IR
- developing theoretical results and comprehensive numerical experiments for LOG-DMS

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Thank you for your attention!

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